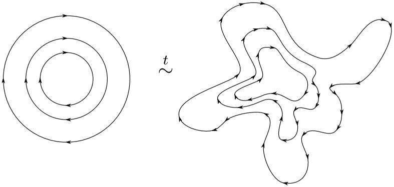
On Topological Equivalence in Linear Quadratic Optimal Control



15th International Young Researchers Workshop on Geometry, Mechanics, and Control — November 30, 2020,

Wouter Jongeneel, Daniel Kuhn

École polytechnique fédérale de Lausanne | RAO
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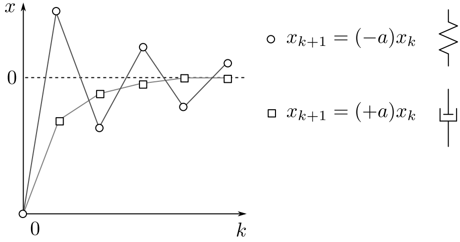
Simple Example..

Given some $a \in (0, 1)$, compare the *behaviour* of a spring and a damper:



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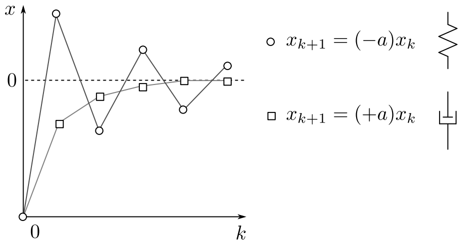

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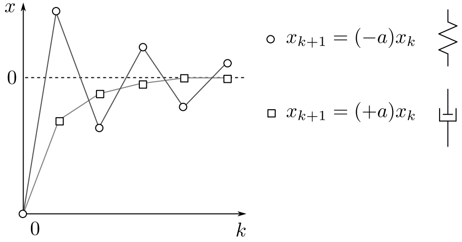

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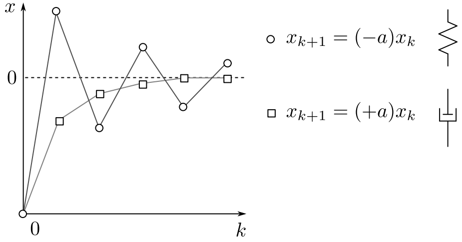

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Regardless of choice of coordinates, we see a structural difference.

Structure in Optimal Control

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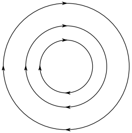
We will observe structural equivalences (or the lack thereoff) in optimal control problems:

$$\arg\min_{f \in \mathcal{F}} J_1(f) \sim \arg\min_{f \in \mathcal{F}} J_2(f) \quad \forall J_1, J_2 \in \mathcal{J}.$$

Topological Equivalence (1/2)

We speak of *topological equivalence*, denoted $\overset{t}{\sim}$, when two phase portraits are homeomorphic and agree on the direction of time.



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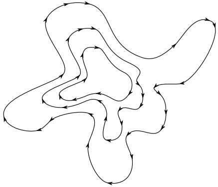

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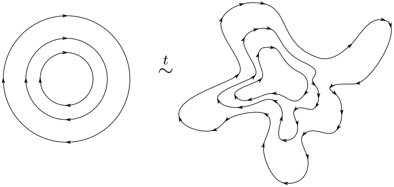

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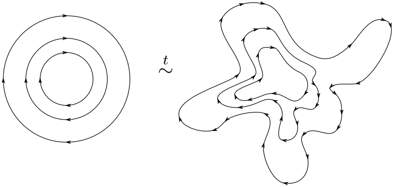

- Hard in continuous-time, since for $\dot{x} = f(x)$ and $\dot{y} = g(y)$ one needs to work with their *flows*.
- In discrete-time, for $x_{k+1} = f(x_k)$, $y_{k+1} = g(y_k)$ seek φ such that $y = \varphi(x)$ relates trajectories, that is, $f = \varphi^{-1} \circ g \circ \varphi$ must hold.

Topological Equivalence (2/2)

Definition (Topological Equivalence): *Two endomorphisms $f : \mathcal{V} \to \mathcal{V}$ and $g : \mathcal{W} \to \mathcal{W}$ over topological vector spaces $\mathcal{V}$ and $\mathcal{W}$ are topologically equivalent (conjugate), denoted $f \overset{t}{\sim} g$, if and only if there exists a homeomorphism $\varphi : \mathcal{V} \to \mathcal{W}$ such that $g \circ \varphi = \varphi \circ f$, that is, the diagram*

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commutes.

$f \overset{t}{\sim} g$ **if a C^0 change of coordinates relates their orbits.**

The Scalar Setting, 7 classes

Consider for some scalar a and b

$$x \mapsto f(x),\ f(x) := ax, \quad y \mapsto g(y),\ g(y) := by.$$

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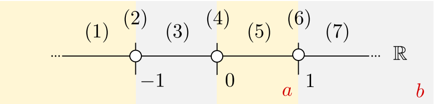

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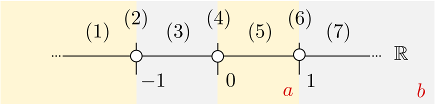

Proposition (Topological Equivalence of Scalar Systems [Kuiper and Robbin 1973, Proposition 1.5]): *Let a and b be members of the same class in $\mathbb{R}$ (see Figure), then $g = \varphi \circ f \circ \varphi^{-1}$ for*

$$\varphi(x) = x|x|^{c-1}, \qquad c = \log(|b|)/\log(|a|).$$

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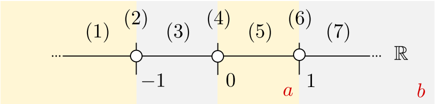

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So for example $f(x) = 2x$ and $g(y) = 8y$ are in class (7) and related by the C^ω map $\varphi(x) = x^3$ while $\varphi^{-1} = x^{1/3}$, which is merely C^0 over $\mathbb{R}$.

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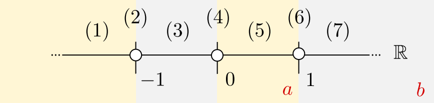

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Similarity transformations would yield a continuum of systems!

A Remark

Building upon Poincaré, Birkhoff, ... The concept of topological equivalence shows, for example, up in the Hartman-Grobman theorem and is key in bifurcation theory. The hope is to study a *finite* amount of classes instead of a continuum.

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Is this true? "*... the classification of all phase portraits on a given manifold, ... up to equivalence under homeomorphisms ... Although some results have been obtained ... it became clear rather early that this program was too ambitious.*" Abraham and Marsden 1978.

For Today

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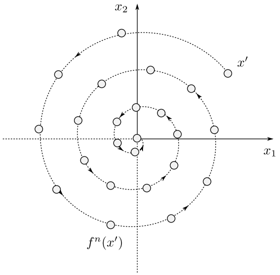
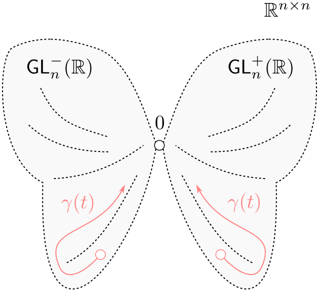
Better yet, focus on the *automorphic* part of $f(x)$,we will look at *invertible* maps $f(x) = Fx$.



The Real General Linear Group

Let $\mathsf{GL}_n(\mathbb{R}) := \{A \in \mathbb{R}^{n\times n} : \det(A) \neq 0\}$, with

- n^2-dim C^ω manifold, dense in $\mathbb{R}^{n\times n}$
- $\mathsf{GL}_n(\mathbb{R}) = \mathsf{GL}_n^+(\mathbb{R}) \overset{\circ}{\bigcup} \mathsf{GL}_n^-(\mathbb{R})$, *e.g.*, $\det(X) > 0 \iff X \in \mathsf{GL}_n^+(\mathbb{R})$.
- Exponential map not surjective.
- $\mathsf{GL}_n^{(i)}(\mathbb{R}), \forall i \in \{+,-\}$ path-connected, yet, not simply-connected.



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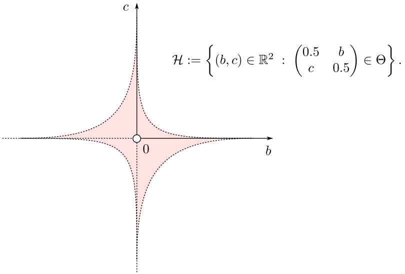

Let $\Theta := \{\theta \in \mathbb{R}^{n \times n} \;:\; \rho(\theta) < 1\}$ be the set of *asymptotically stable* matrices. Θ is a semi-algebraic non-convex (star-convex) set.

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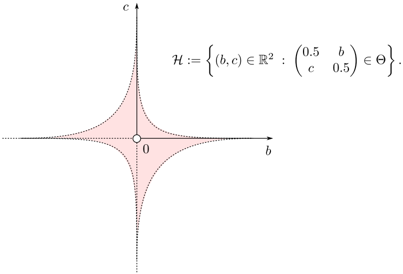

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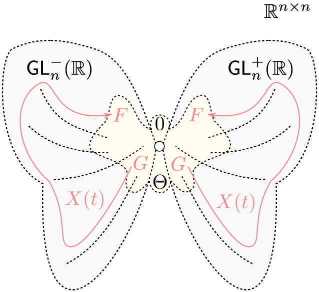
We will usually talk about $F \in \mathsf{GL}_n(\mathbb{R}) \cap \Theta$, that is $\lim_{n\to\infty} f^n(x) = 0 \ \forall x$.

Characterizing Equivalence

Theorem (Topological Equivalence of Asymptotically Stable Systems [Robinson 1995, Theorem 9.2 page 117]): *Let $f(x) := Fx$ and $g(y) := Gy$ be asymptotically stable linear automorphisms on $\mathbb{R}^n$. Moreover, let $X(t)$ parametrize a path in $\mathsf{GL}_n(\mathbb{R})$, continuously depending on $t \in [0, 1]$, such that $X(0) = F$ and $X(1) = G$, then, $f \overset{t}{\sim} g$.*



Linear Quadratic Optimal Control (1/3)

The general (discrete-time) Linear Quadratic Optimal Control problem is given by

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\underset{\{u_k\}_{k\geq 0}}{\text{minimize}} \quad & \sum_{k=0}^{\infty} \begin{pmatrix} x_k \\ u_k \end{pmatrix}^\mathsf{T} \begin{pmatrix} Q & S \\ S^\mathsf{T} & R \end{pmatrix} \begin{pmatrix} x_k \\ u_k \end{pmatrix} \\
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If (A, B, C), $C^{\mathsf{T}}C := Q$, is a *minimal* realization, $R \succ 0$, then, $u_k^{\star} = K^{\star}x_k$

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P &= Q + A^{\mathsf{T}} \left(P - PB(R + B^{\mathsf{T}}PB)^{-1}B^{\mathsf{T}}P\right) A, \\
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"*Tuning*" the pair (Q, R)?

If in

$$c(x,u) = \begin{pmatrix} x \\ u \end{pmatrix}^{\mathsf{T}} \begin{pmatrix} Q & S \\ S^{\mathsf{T}} & R \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix}.$$

$S \neq 0$ "diagonalize" via $v := R^{-1}S^{\mathsf{T}}x + u$, $Q' = Q - SR^{-1}S^{\mathsf{T}}$ and $A' := A - BR^{-1}S^{\mathsf{T}}$, obtain the standard LQ problem:

$$\begin{aligned} \underset{\{v_t\}_{t\geq 0}}{\text{minimize}} \quad & \sum_{k=0}^{\infty} \underbrace{x_k^{\mathsf{T}} Q' x_k + v_k R v_k}_{\text{stage cost } c'(x_k, v_k)} \\ \text{subject to} \quad & \underbrace{x_{k+1} = A' x_k + B v_k,}_{x_k \mapsto \sigma'(x_k, v_k)} \quad x_0 = x' \end{aligned}$$

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For a sensible solution we need the same conditions as before, yet, additionally $\mathrm{rank}(Q) = \mathrm{rank}(Q')$, otherwise we solve a different problem.

Let $\underbrace{(A,B)}_{x\mapsto\sigma(x,u)}$ be a stabilizable pair, then regarding the cost, we look at the set

$$\mathcal{C}(\sigma) := \left\{ (Q,R,S) \in \mathcal{S}^n_{\succeq 0} \times \mathcal{S}^m_{\succ 0} \times \mathbb{R}^{n\times m} : \begin{array}{l} \operatorname{rank}(Q) = \operatorname{rank}(Q'), \\ \exists C \in \mathbb{R}^{p\times n} : C^{\mathsf{T}}C = Q, \\ (A,C) \text{ detectable} \end{array} \right\}.$$

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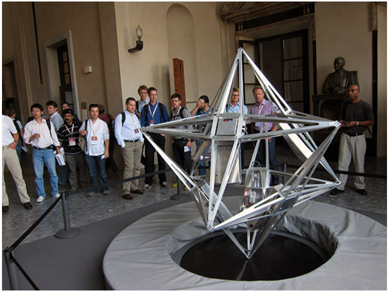
"*Tuning*" $(Q, R, S) \in \mathcal{C}(\sigma)$? Example: Trimpe and D'Andrea 2012, their motivation was to penalize $\|u_k - u_{k-1}\|$



idsc.ethz.ch/research-dandrea/research-projects/archive/balancing-cube

Topological Perspective on Tuning (1/2)

Introduce a orientation-dependent version of $\mathcal{C}(\sigma)$ Given a $\sigma \in \Sigma$ such that $A \in \mathsf{GL}(n, \mathbb{R})$ and define $\mathcal{C}^{(i)}(\sigma)$, $(i) \in \{+, -\}$ by

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Proof sketch: $x \mapsto \sigma_j^{\star}(x)$ is of the form $\Lambda_j^{-1} A_j' x$ with $\Lambda_j \in \mathsf{GL}_n^{+}(\mathbb{R})$.

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- Consequences for Inverse Optimal Control,given any $K' \in \mathbb{R}^{m\times n}$:

$$x^\mathsf{T}(K-K')^\mathsf{T}(K-K')x = \begin{pmatrix} x \\ Kx \end{pmatrix}^\mathsf{T} \begin{pmatrix} K'^\mathsf{T}K' & -K'^\mathsf{T} \\ -K' & I_m \end{pmatrix} \begin{pmatrix} x \\ Kx \end{pmatrix}.$$

The Real Symplectic Group

A different point of view.

Let the billinear form $\omega : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R}$ be defined as $\omega(x, y) = x^\mathsf{T} \Omega y$ for $\Omega \in \mathbb{R}^{2n \times 2n}$ as given by

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Then, define the real *Symplectic group* by
$\mathsf{Sp}(2n, \mathbb{R}) := \{M \in \mathbb{R}^{2n \times 2n} : M^\mathsf{T} \Omega M = \Omega\}$.

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Define $M \in \mathsf{Sp}(2n, \mathbb{R})$ (Hamiltonian) by

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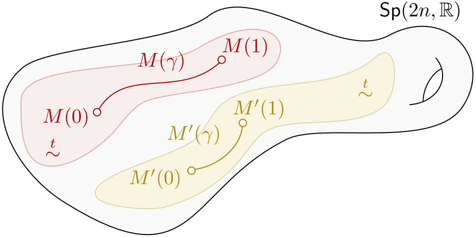
Symplectic Perspective (2/2)

$$M(\gamma) := \begin{bmatrix} A'(\gamma) + BR(\gamma)^{-1}B^{\mathsf{T}}A'(\gamma)^{-\mathsf{T}}Q'(\gamma) & -BR(\gamma)^{-1}B^{\mathsf{T}}A'(\gamma)^{-\mathsf{T}} \\ -A'(\gamma)^{-\mathsf{T}}Q'(\gamma) & A'(\gamma)^{-\mathsf{T}} \end{bmatrix}.$$



The asymptotically stable automorphic case.

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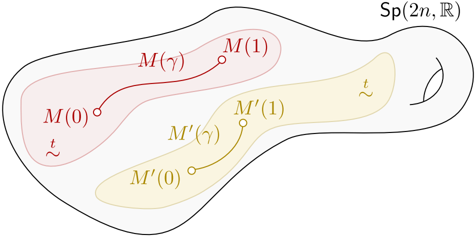

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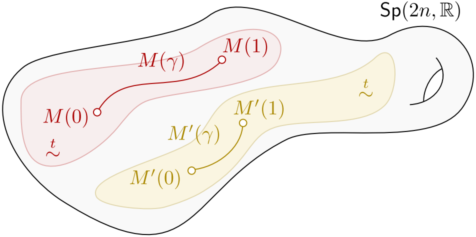

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Can this be generalized to other Hamiltonians?

Example: Bifurcation by tuning

Consider the LQR problem for $B = I_2$, $R = I_2$, $Q = 10 \cdot I_2$ and

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad S(\varepsilon) = \begin{pmatrix} \varepsilon & 0 \\ 0 & -\varepsilon \end{pmatrix}, \quad \varepsilon \in \mathbb{R}_{\geq 0}.$$

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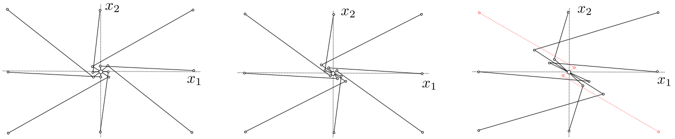

(a) ($\varepsilon = 0$) (b) ($\varepsilon = 0.5$) (c) ($\varepsilon = 2$)

Figure: A few closed-loop trajectories as a function of $S(\varepsilon)$.

Example: Poincaré maps (1/3)

In continuous-time: $\dot{x} = Ax$, $x(0) = x'$ leads to $x(t) = e^{At}x'$. Sampling $(\mu > 0)$ then leads to $x \mapsto e^{A\mu}x$: *orientation preserving* map.

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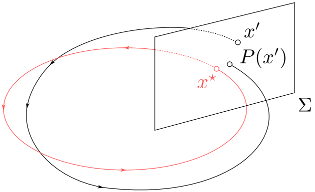

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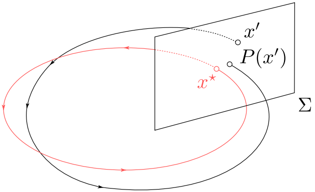

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What about controlled Poincaré maps?

Example: Poincaré maps (2/3)

Consider the affine dynamical control system given by:

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Since $1 = P(1, 0)$, linearize around $r = 1$, that is, for $\xi := r - 1$ obtain the local linear model:

$$\xi_{k+1} = e^{4\pi}\xi_k + \left(\frac{1}{2}e^{4\pi} - \frac{1}{2}\right) u_k =: A_\xi \xi_k + B_\xi u_k.$$

Example: Poincaré maps (3/3)

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(a) Trajectories under $K_\xi^\star$ (LQR). (b) Trajectories under $\widetilde{K}$.

Figure: One controlled cycle for both $K_\xi^\star$ and $\widetilde{K}$.

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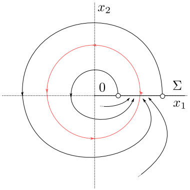

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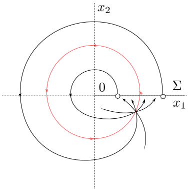

(b) Trajectories under $\widetilde{K}$.

Figure: One controlled cycle for both $K_\xi^\star$ and $\widetilde{K}$.

Periodic version of damper vs. spring.

More family members

The result extends to the whole *family* of Linear Quadratic (LQ) optimal control problems, that is, dynamic games, LEQR, H_∞-control, etc.

After the Ending

Based on:

N.H. Kuiper and J.W. Robbin (1973). "Topological Classification of Linear Endomorphisms". In: *Inventiones math. Springer Verlag* 19, pp. 83–106

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[JK20] Wouter Jongeneel and Daniel Kuhn (2020), *On Topological Equivalence in Linear Quadratic Optimal Control* http://wjongeneel.nl/pub/TopoEquiv.pdf.

For more information, see wjongeneel.nl or rao.epfl.ch