A generalized global Hartman-Grobman theorem for asymptotically stable semiflows

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***Abstract*— Recently, Kvalheim and Sontag provided a generalized global Hartman-Grobman theorem for equilibria under asymptotically stable continuous vector fields. By leveraging topological properties of Lyapunov functions, their theorem works without assuming hyperbolicity. We extend their theorem to a class of possibly discontinuous vector fields, in particular, to vector fields generating asymptotically stable semiflows.**

***Index Terms*— asymptotic stability, Hartman-Grobman linearization, Koopman theory, semiflow**

I. Introduction

Linearization, in all its forms, remains one of the most powerful techniques to handle nonlinear sytems, *e.g.*, there is machinery that is widely, rigorously and easily applicable. Unfortunately, one can typically only say something locally. Therefore, within the broader field of linearization techniques, Koopman operator theory is of great interest as the analysis need not be local. However, in a recent survey on modern Koopman theory we find the following: "*..., obtaining finite-dimensional coordinate systems and embeddings in which the dynamics appear approximately linear remains a central open challenge.*" [1, p. 1] and "*..., there is little hope for global coordinate maps of this sort.*" [1, p. 4].

Clearly, these comments relate to the desire of obtaining tools akin to the celebrated Hartman-Grobman theorem (*e.g.*, see [2]); ideally, tools that are applicable globally and without relying on hyperbolicity. Indeed, a recent global extension of the Hartman-Grobman theorem by Kvalheim and Sontag exploits stability, and in particular Lyapunov theory, to overcome the restrictive reliance on hyperbolicity *cf.* [3]–[6]. Here, the intuition is that the Lyapunov function replaces the Morse function. We remark that in the context of topological embeddings (*i.e.*, not necessarily homeomorphisms), hyperbolicity has been relaxed before, also by exploiting stability, *e.g.*, see [7, Cor. 4]. Now, elaborating on [6], we can construct a similar result when the vector field is not even continuous at the equilibrium. As in [6], we build upon the topological results in [8] and [9].

To illustrate our setting, the running example we have in mind is a normalized vector field on $\mathbb{R}^n$ of the form

$$\dot{x} = \begin{cases} -\dfrac{x}{\|x\|_2} & x \in \mathbb{R}^n \setminus \{0\} \\ 0 & \text{otherwise,} \end{cases} \tag{1}$$

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where the solutions will be understood *in the sense of Filippov*[1], with $\mathcal{F}$ denoting the Filippov operator [11, p. 85]. For instance, the solution corresponding to (1) becomes

$$(t,x) \mapsto \varphi_1^t(x) := \begin{cases} \left(1 - \dfrac{t}{\|x\|_2}\right) x & t \leq \|x\|_2 \\ 0 & \text{otherwise,} \end{cases} \tag{2}$$

which is not a flow, but a *semi*flow. The reason being, $(t,x) \mapsto \varphi_1^t(x)$ need not be well-defined for $t < 0$, *e.g.*, consider $\varphi_1^t(0)$ for any $t < 0$.

Recall, given a topological space M, a continuous map $\varphi : \mathbb{R}_{\geq 0} \times M \to M$ is said to be a *semiflow* when $\varphi^0 = \mathrm{id}_M$ and $\varphi^s \circ \varphi^t = \varphi^{s+t}$ for all $s, t \in \mathbb{R}_{\geq 0}$.

We will study semiflows generated by vector fields and thus we consider sufficiently regular manifolds and not arbitrary topological spaces. Then, in general, given a semiflow φ on some topological manifold M, we speak of a *linearizing* (with respect to φ) homeomorphism $h : M \to \mathbb{R}^n$ when $h \circ \varphi^t = e^{t \cdot A} \circ h$ for all $t \in \mathbb{R}_{\geq 0}$ and some matrix $A \in \mathbb{R}^{n \times n}$ (*i.e.*. φ^t and $e^{t \cdot A}$ are topologically conjugate). Observe that the Hartman-Grobman theorem provides us with a particularly convenient, but *local*, linearization (*e.g.*, A is of the form $\mathrm{D}X(x_*)$ for some C^1 vector field X and hyperbolic equilibrium point x_*).

When it comes to linearizing a system like (1), then, due to the finite-time stability property of such a semiflow, there cannot be a homeomorphism h such that the conjugacy $h \circ \varphi_1^t \circ h^{-1} = e^{-t \cdot I_n}$ holds true for all $(t, h^{-1}(x)) \in \mathrm{dom}(\varphi_1)$, for otherwise $\dot{x} = -x$ would correspond to a *finite*-time stable system. With this observation in mind, in this short note we show to what extent semiflows corresponding to vector fields like (1) can still be "*linearized*". We emphasize that we will not consider a reparametrization of time.

Notation. We let $B(0,r) := \{x \in \mathbb{R}^n : \|x\|_2 < r\}$ be the open ball of radius r, centered at $0 \in \mathbb{R}^n$, with $B(0,r)^c := \mathbb{R}^n \setminus \{B(0,r)\}$ denoting its complement. With $\mathrm{cl}(W)$ and ∂W we denote the topological closure and manifold boundary of W, respectively. The symbol $\simeq_t$ denotes topological equivalence, whereas $\simeq_h$ denotes homotopy equivalence. A continuous function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{K}_\infty$ when $\gamma(0) = 0$, γ is strictly increasing and $\lim_{s \to +\infty} \gamma(s) = +\infty$.

II. Main result

On $\mathbb{R}^n$, the vector fields we consider are possibly set-valued at 0, locally essentially bounded on $\mathbb{R}^n$ and locally Lipschitz on $\mathbb{R}^n \setminus \{0\}$, *e.g.*, like (1). It is known that

[1] See [10] for a comparison of solution frameworks.

under these assumptions, $\mathcal{F}[X]$ is upper semi-continuous and compact, convex valued, which allows for a *smooth* converse Lyapunov theory, *e.g.*, see [12]. Indeed, one can do with significantly less regular vector fields. However, we focus on (1) and keep the presentation simple.

For the appropriate generalization to manifolds, any manifold M we consider is smooth, second countable and Hausdorff (to appeal to Whitney's embedding theory, *e.g.*, see [13, Cor. 6.16]). Then, a set-valued vector field $F : M \rightrightarrows TM$ (*e.g.*, think of $\mathcal{F}[X]$) is said to satisfy the *basic conditions* when it is a locally bounded map, outer-semicontinuous and $F(x)$ is non-empty, compact and convex for all $x \in M$ [14, Def. 5]. These basic conditions suffice for a smooth converse Lyapunov theory [14, Cor. 13] via [15]. In particular, we consider the following class of vector fields.

Assumption II.1 (Vector field regularity on M)**.** *Given a point $x' \in M$, our vector fields are locally essentially bounded on M, possibly set-valued at x' and locally Lipschitz on $M \setminus (\partial M \cup \{x'\})$.*

Suppose that X complies with Assumption II.1, see that after smoothly embedding M into some $\mathbb{R}^k$, the vector field in new coordinates still satisfies the conditions of Assumption II.1. Then, it is known that under Assumption II.1, the differential inclusion that corresponds to the Filippov operator (*i.e.*, applied after embedding M), satisfies the basic conditions from above. For simplicity of exposition, however, we will now directly work with embedded submanifolds $M \subseteq \mathbb{R}^k$ (generalizations are of course immediate).

Also, see that Assumption II.1 only allows for mildly discontinuous vector fields X akin to (1), *i.e.*, the local boundedness is only relevant for a neighbourhood of x'.

Let X be a vector field on an embedded submanifold $M \subseteq \mathbb{R}^k$ that complies with Assumption II.1, then, from now on, the corresponding (Filippov) *solutions* $(t,p) \to \varphi^t(p)$ to the differential equation $\dot{x} = X(x)$, are understood to be absolutely continuous in t, on compact intervals $\mathcal{I}$, and such that they satisfy

$$\frac{\mathrm{d}}{\mathrm{d}s}\varphi^s(p)\bigg|_{s=t} \in \mathcal{F}[X](\varphi^t(p)) \text{ for a.e. } t \in \mathcal{I}.$$

We study (strong) (global) asymptotic stability of these solutions and point to [12, Def. 2.1] and [14, Sec. 2] for further details. Note in particular, that under Assumption II.1, we will have uniqueness of solutions if we set x' to be the attractor, this, thanks to stability.

Our main result consists of two cases, with case *(I)* being reminiscent of (1) on all of $\mathbb{R}^n$ whereas case *(II)* captures, for instance, (1) defined on a *bounded* domain (*i.e.*, this is why we consider the possibility of $\partial M \neq \varnothing$ in Assumption II.1).

Before stating the result, let us clarify the inherent difficulties with semiflows. Let B be the basin of attraction of $x_* \in M$ under some flow φ. In [6], the authors construct their linearizing homeomorphism by exploiting that $B \setminus \{x_*\} \simeq_t \mathbb{R} \times \mathbb{S}^{n-1}$, that is, they exploit that $\mathbb{R} \times B \subseteq \mathrm{dom}(\varphi)$. In the case of semiflows, one might define the map $T_*^+ : B \setminus \{x_*\} \to \mathbb{R}_{\geq 0}$ via $T_*^+(x) := \inf\{t \geq 0 : \varphi^t(x) = x_*\}$ and hope to use $(-\infty, T_*^+(x))$ instead of $(-\infty, +\infty)$. A problem is, even under Assumption II.1, T_*^+ cannot be guaranteed to be continuous. For instance, consider the vector field X on the plane, defined on $\mathbb{R}^2 \setminus \{0\}$ through

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix} = X(x) := \begin{pmatrix}-x_1\\ -\dfrac{x_2}{|x_2| + x_1^2}\end{pmatrix} \tag{3}$$

with $X(0) := 0$. In this case, T_*^+ is discontinuous at $\{(0, x_2) : x_2 \neq 0\}$ (jumping from $|x_2|$ to $+\infty$). Hence, in the following proof we will look at the smallest $t \geq 0$ such that $\varphi^t(x)$ enters a *neighbourhood* of x_* instead.

In the following, *closed* refers to closed in the subspace topology, not compact and without boundary.

Theorem II.2 (A global Hartman-Grobman theorem, without hyperbolicity, for asymptotically stable semiflows)**.** *For $M \subseteq \mathbb{R}^k$ a closed, smooth, n-dimensional embedded submanifold, let $x_* \in M$ be asymptotically stable, under a semiflow φ (i.e., a Filippov solution) generated by a vector field that satisfies Assumption II.1 for $x' := x_*$ and let $B \subseteq M$ be the corresponding basin of attraction.*

(I) Suppose that $(-\infty, 0] \times B \setminus \{x_*\} \subseteq \mathrm{dom}(\varphi)$, then, for any $r > 0$ there is a homeomorphism $h_r : B \to \mathbb{R}^n$ and a $\gamma_r \in \mathcal{K}_\infty$ such that for all $y \in B(0,r)^c$

$$h_r \circ \varphi^t|_B \circ h_r^{-1}\big|_{B(0,r)^c}(y) = e^{-t \cdot I_n}\big|_{B(0,r)^c}(y) \tag{4}$$

holds for all t such that $0 \leq t \leq \gamma_r(\|y\|_2 - r)$.

(II) Suppose that $(-\infty, 0] \times B \setminus \{x_*\} \nsubseteq \mathrm{dom}(\varphi)$, then, for any $r > 0$ there is a homeomorphism $h_r : B \to \mathbb{R}^n$, a $\gamma_r \in \mathcal{K}_\infty$ and a $R > r$ such that for all $y \in \mathrm{cl}(B(0,R)) \setminus B(0,r)$, (4) *holds for all t such that $0 \leq t \leq \gamma_r(\|y\|_2 - r)$.*

Proof. The proof is an extension of that of [6, Thm. 2]. We follow their arguments and notation to a large extent.

We start with *(I)*. By asymptotic stability of x_* and Assumption II.1, there is a C^∞ Lyapunov $V : B \setminus \{x_*\} \to \mathbb{R}_{\geq 0}$ corresponding to the pair (x_*, φ) [12], [14], [15]. Now consider $V^{-1}(\varepsilon)$ for some $\varepsilon > 0$. As $V^{-1}(\varepsilon) \simeq_h \mathbb{S}^{n-1}$ [8], then, by the resolution of the *topological* Poincaré conjecture, there is always a homeomorphism $P : V^{-1}(\varepsilon) \to \mathbb{S}^{n-1}$.

Let $L_\varepsilon := V^{-1}(\varepsilon)$ and define the map $T_\varepsilon^+ : B \setminus \{x_*\} \to \mathbb{R}_{\geq 0}$ through $T_\varepsilon^+(x) := \inf\{t \geq 0 : \varphi^t(x) \in L_\varepsilon\}$. Following similar arguments as in [16, Thm. 5], it follows that T_ε^+ is continuous on $B \setminus \{x_*\}$. As discussed above, we cannot simply set $\varepsilon := 0$. Now define $U_\varepsilon := V^{-1}([0, \varepsilon))$ and

$$W := \bigcup_{x \in L_\varepsilon} (-\infty, 0] \times \{x\} \subseteq \mathbb{R} \times L_\varepsilon,$$

then, thanks to continuity of T_ε^+ and the standing assumptions (*i.e.*, away from x_* we can move backwards indefinitely, but forwards we might converge in finite-time) it follows that $(t,x) \mapsto \varphi^t(x)$ defines a homeomorphism from W to $B \setminus U_\varepsilon$, with inverse $g = (\tau, \rho) : B \setminus U_\varepsilon \to W$, *i.e.*, $\rho(x) \in L_\varepsilon$ and $\varphi^{\tau(x)}(\rho(x)) = x$. However, it will be more convenient to define the inverse g through $\varphi^{-\tau(x)}(\rho(x)) = x$ so that $\tau(x) \to -T_\varepsilon^+(\rho(x)) = 0^-$ for $B \setminus U_\varepsilon \ni x \to L_\varepsilon$.

The homeomorphism h from [6], as defined through $x \mapsto h(x) := e^{\tau(x)}P(\rho(x))$, was designed for flows and relies on $\tau(x) \to -\infty$ for $x \to x_*$. In particular, their domain of τ is $B \setminus \{x_*\}$ and not $B \setminus U_\varepsilon$. Thus, we need to rescale. In particular, consider the homeomorphism $x \mapsto h'(x) := e^{\tau'(x)}P(\rho'(x))$ from B to $\mathbb{R}^n$, with τ' defined on $B \setminus \{x_*\}$ through

$$\tau'(x) := \begin{cases} \tau(x) & x \in B \setminus U_\varepsilon \\ \ln\left(\dfrac{V(x)}{\varepsilon}\right) & \text{otherwise.} \end{cases} \tag{5}$$

Now, define the map $T^-_\varepsilon : (L_\varepsilon \cup U_\varepsilon) \setminus \{x_*\} \to \mathbb{R}_{\geq 0}$ through $T^-_\varepsilon(x) = \inf\{t \geq 0 : \varphi^{-t}(x) \in L_\varepsilon\}$, and eventually ρ' on $B \setminus \{x_*\}$ via

$$\rho'(x) := \begin{cases} \rho(x) & x \in B \setminus U_\varepsilon \\ \varphi^{-T^-_\varepsilon(x)}(x) & \text{otherwise.} \end{cases}$$

At last, set $h'(x_*) := 0$. Regarding continuity, observe that for $x \in L_\varepsilon$, we have $\tau(x) = 0$ and for $U_\varepsilon \ni x \to L_\varepsilon$ we have $\tau(x) \to 0^-$. Moreover, for $x \to x_*$ we have that $\tau'(x) \to -\infty$ and thus $e^{\tau'(x)} \to 0$ such that thanks to compactness of L_ε we have $h(x) \to 0$. Continuity of T^-_ε follows again from [16, Thm. 5] and continuity of h'^{-1} follows, for instance, from an open mapping argument (*e.g.*, see [13, Thm. A.38]).

Following [6], see that for any $x \in B \setminus U_\varepsilon$ we have $\rho(\varphi^t(x)) = \rho(x)$ and for sufficiently small $t \geq 0$ we have $\tau(\varphi^t(x)) = \tau(x) - t$ (*e.g.*, see that $\varphi^t(x) = \varphi^t(\varphi^{-\tau(x)}(\rho(x))) = \varphi^{-\tau(\varphi^t(x))}(\rho(x))$).

Specifically, see that $\tau'(\varphi^t(x)) = \tau(x) - t$ for $x \in B \setminus U_\varepsilon$ and $t \leq T^+_\varepsilon(x)$ with $T^+_\varepsilon(x) > 0$ for $x \in B \setminus (L_\varepsilon \cup U_\varepsilon)$.

Now, let $y := h'(x)$, then $x \in L_\varepsilon \implies y = h'(x) \in \mathbb{S}^{n-1}$. Hence, (5) implies that outside of $B(0,1)$, the semiflow of the conjugate system, that is, $(t,y) \mapsto h' \circ \varphi^t|_B \circ h'^{-1}(y)$, is equivalent to $(t,y) \mapsto e^{-t}y$, for $t \leq T^+_\varepsilon(x) = T^+_\varepsilon(h'^{-1}(y))$ and $y \in B(0,1)^c$. To recover h_r of the theorem, simply scale h' by r to scale the radius of the ball (recall that $P : L_\varepsilon \to \mathbb{S}^{n-1}$).

At last, define $\gamma_r : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ through

$$s \mapsto \gamma_r(s) := \inf_{y \in \mathbb{R}^n} \left\{ T^+_\varepsilon \circ h_r^{-1}|_{B(0,r)^c}(y) : \|y\|_2 = s + r \right\}.$$

It follows from, for instance, Berge's maximum theorem [17, p. 115] that γ_r is continuous and from the fact that on $B \setminus \{x_*\}$, $\varphi^t(x)$ extends indefinitely backwards, that $\gamma_r \to +\infty$ for $\|y\|_2 \to +\infty$. Hence, as by properties of φ, V and T^+_ε, γ_r is strictly monotone, we have that $\gamma_r \in \mathcal{K}_\infty$.

To continue with *(II)*, the arguments are almost identical, yet, now we need to carefully deform $B \setminus \{x_*\}$ around all its "*boundaries*", not only around x_*.

To that end, pick some $C > \varepsilon$ and define $L_C := V^{-1}(C)$ and the map $T^-_C : B \setminus \{x_*\} \to \mathbb{R}_{\geq 0}$ through $T^-_C(x) = \inf_t\{t \geq 0 : \varphi^{-t}(x) \in L_C\}$. Now, if we define

$$W_C := \bigcup_{x \in L_\varepsilon} [-T^-_C(x), 0] \times \{x\} \subseteq \mathbb{R} \times L_\varepsilon,$$

then, $(t,x) \mapsto \varphi^t(x)$ yields a homeomorphism from W_C to $(L_C \cup U_C) \setminus U_\varepsilon \subset B \setminus \{x_*\}$, for $U_C := V^{-1}([0,C))$. To proceed, we will effectively "*linearize*" on this compact set $K_{\varepsilon,C} := (L_C \cup U_C) \setminus U_\varepsilon = V^{-1}([\varepsilon, C])$, accomodated by appropriate deformations on the remaining space.

With this construction in mind, consider now the homeomorphism $x \mapsto h_C(x) := e^{\tau_C(x)}P(\rho_C(x))$ from B to $\mathbb{R}^n$, with again $h_C(x_*) := 0$, $\rho_C := \rho'$ and with τ_C defined on $B \setminus \{x_*\}$ through

$$\tau_C(x) := \begin{cases} \tau(x) + (V(x) - V(C)) & x \in B \setminus K_{\varepsilon,C} \\ \tau(x) & x \in K_{\varepsilon,C} \\ \ln\left(\dfrac{V(x)}{\varepsilon}\right) & \text{otherwise.} \end{cases}$$

Note that by coercivity of V, we have $\tau_C(x) \to +\infty$ for $x \to \lim_{t \to -\infty} \varphi^t(x)$, with $x \in B \setminus \{x_*\}$.

Again, we can multiply h_C by r to rescale. In this case, R is given by $\inf_{x \in L_C} r \cdot e^{\tau_C(x)}$, which is attained due to compactness of L_C and continuity of τ_C. As $\tau_C(x) = 0$ for $x \in L_\varepsilon$ and $C > \varepsilon$, we have that $R > r$. Note, this exponential term defining R is exactly what one should expect, given that we do not reparametrize time (*e.g.*, we should have $r = e^{-T}R$ for some appropriate $T > 0$).

At last, we can also employ T^+_ε and γ_r again, subject to constraining their domain appropriately. $\square$

Of course, if φ happens to be a flow, Theorem II.2 is also true, but slightly conservative as there is no need to construct $B(0,r)$ and the like *cf.* [6, Thm. 2].

III. Example

To exemplify Theorem II.2, we discuss case *(I)* (as case *(II)* follows as a corollary). For the notation, we refer to Theorem II.2 and its proof.

Specifically, regarding (1) on $\mathbb{R}^n$, note that $x \mapsto V(x) := \frac{1}{2}\|x\|_2^2$ checks out as a Lyapunov function[2] so that $B = \mathbb{R}^n$ and $x_* = 0$. Indeed $V^{-1}(\varepsilon) = \{x \in \mathbb{R}^n : \|x\|_2 = \sqrt{2\varepsilon}\}$, so we simply select $\varepsilon := \frac{1}{2}$ such that $P : V^{-1}(\varepsilon) \to \mathbb{S}^{n-1}$ becomes the identity map on $\mathbb{S}^{n-1}$.

As, $T^-_\varepsilon(x) = 1 - \|x\|_2$, we get that $x \mapsto \rho'(x) := x/\|x\|_2$ for all $x \in B \setminus \{x_*\}$. Then since $\varphi^{-\tau(x)}(\rho(x)) = x$ we get that $\tau'(x) = \|x\|_2 - 1$ on $B \setminus U_\varepsilon$ and $\tau'(x) = \ln(\|x\|_2^2/2\varepsilon)$ on $U_\varepsilon \setminus \{x_*\}$.

Now, the homeomorphism $h_r : \mathbb{R}^n \to \mathbb{R}^n$ from Theorem II.2 is explicitly given as

$$h_r(x) := \begin{cases} r \cdot e^{\tau'(x)} \dfrac{x}{\|x\|_2} & x \in \mathbb{R}^n \setminus \{0\} \\ 0 & \text{otherwise.} \end{cases}$$

Indeed, for $x \to x_*$ we have that $e^{\tau'(x)} \to 0$. We can also explicitly state the inverse $h_r^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ as

$$h_r^{-1}(y) := \begin{cases} \alpha_r(y) \dfrac{y}{\|y\|_2} & y \in \mathbb{R}^n \setminus \{0\} \\ 0 & \text{otherwise} \end{cases}$$

[2]Indeed, this Lyapunov function even certifies finite-time stability as $\dot{V} \leq -\frac{1}{2}V^\alpha$ for $\alpha = \frac{1}{2}$, *e.g.*, see [18].

with α_r defined on $\mathbb{R}^n \setminus \{0\}$ through

$$\alpha_r(y) := \begin{cases} \ln\left(\frac{\|y\|_2}{r}\right) + 1 & y \in B(0,r)^c \\ \left(\frac{\|y\|_2}{r}\right)^{1/2} & \text{otherwise.} \end{cases}$$

Now we have the ingredients to explicitly compute $h_r \circ \varphi^t \circ h_r^{-1}$. We do this for (i) $y \in B(0,r)^c$ and (ii) for $y \in B(0,r)$.

(i) First, suppose that we have $y \in B(0,r)^c$ such that $\|y\|_2 = \delta r$ for some $\delta \geq 1$. It follows that $\alpha_r(y) = 1 + \ln(\delta)$ and thus $x := h_r^{-1}(y) = (1+\ln(\delta))y/\|y\|_2$. Recall that $\|\varphi^t(x')\|_2 = \|x'\|_2 - t$, for any $x' \in \mathbb{R}^n \setminus \{0\}$ and $t \leq \|x'\|_2$ and thus, when looking at the definition of τ', we have that $\varphi^t(x) \in B(0,1)^c$ for $t \leq \ln(\delta)$. Putting it all together, we get $h_r \circ \varphi^t \circ h_r^{-1}(y) = r\delta e^{-t} y/\|y\|_2$ for $t \leq \ln(\delta)$. Therefore, under the homeomorphism h_r, once we start in $B(0,r)^c$, we flow towards $B(0,r)$ along the canonical dynamics $\dot{x} = -x$.

We add that a simple computation shows that $\gamma_r \in \mathcal{K}_\infty$, as constructed in the proof of Theorem II.2, becomes

$$s \mapsto \gamma_r(s) := \ln\left(\frac{s+r}{r}\right).$$

Indeed, then $\gamma_r(\|y\|_2 - r) = \ln(\delta)$.

(ii) Now suppose we start within $B(0,r)$, say $\|y\|_2 = \theta r$ with $\theta \in (0,1)$ ($\theta = 0$ is not very interesting). To comply with topologial conjugacies, we must recover finite-time stability around the stable equilibrium. See that $a_r(y;\theta) = \sqrt{\theta}$ and thus $x := h_r^{-1}(y) = \sqrt{\theta} y/\|y\|_2$ with $\|x\|_2 < 1$. This means that we look at the second case of τ' and

$$h_r \circ \phi^t \circ h_r^{-1}(y) = r\left(\sqrt{\theta} - t\right)^2 \frac{y}{\|y\|_2} \tag{6}$$

for $t \leq \sqrt{\theta}$. Thus, finite-time stability is preserved, as it should. Comparing (6) to (2), see that apparent scaling is due to our *choice* of Lyapunov function.

IV. Discussion

There are several ways to prove Theorem II.2 and similar results. We did not consider a reparametrization of time, nor the simplest form the dynamics could have outside of the domain of linearization. Plus, for simplicity, we have focused on Euclidean balls (*e.g.*, $B(0,r)$), which could be improved.

In spirit, Theorem II.2 can also be understood as a non-local (and non-constant) version of flow-box (or straightening-out) results *cf.* [19, Thm. 2.26]. Better yet, one may interpret the above as a notion of a *practical linearization*, akin to practical stability [20, §25] that is.

Results like these not only reinforce the Koopman operator framework, but also the "*hybridization*" of topological dynamical systems theory, *e.g.*, see [21], [22]. Moreover, these results are closely related to recent studies [23], [24] into the topology of stable systems as decompositions like (4) allow for studying spaces of stable systems through homeomorphism groups, *e.g.*, see [23, Prop. 3.2].

Regarding future work, we are especially interested in generalizations beyond equilibrium points, that is, to general compact attractors.

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