

Preorders of multiple Lyapunov functions can always be lifted to simulation relations[★]

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Abstract: To compare multiple Lyapunov functions in the context of switched systems, several preorders have been introduced. Unfortunately, these preorders are typically intratable. For a handful of instances, explicit lifts—of graphs that capture the multiple Lyapunov functions—have been constructed such that the preorder relation corresponds to a graphical simulation relation, after the lift. In this note we show that such a lift always exists, for any preorder.

Keywords: Lyapunov methods, Stability and stabilization of hybrid systems

1. INTRODUCTION

Switched systems (Liberzon, 2003), are a popular topic of study since these systems naturally emerge when modelling physical phenomena, plus, they allow for overcoming several topological obstructions and function as a benchmark setting for a more general hybrid systems theory (Goebel et al., 2012; Sanfelice, 2020; Jongeneel and Moulay, 2023). In particular, understanding *stability* of switched systems is important; interestingly, fully tackling the linear case is still open.

1.1 Stability of switched (linear) systems

We are interested in studying the stability of $0 \in \mathbb{R}^n$ under a discrete-time switched system of the form

$$x(k+1) = f_{\sigma(k)}(x(k)), \quad k \in \mathbb{N}_{\geq 0}, \quad x(0) = x_0, \quad (1)$$

where $\sigma : \mathbb{N}_{\geq 0} \rightarrow \Sigma$ is the switching signal, for Σ a finite alphabet that parametrizes a set of maps $\mathcal{F} = \{f_i \in C^0(\mathbb{R}^n; \mathbb{R}^n) \mid f_i(0) = 0, i \in \Sigma, \}$ (the modes). In what follows we assume to have no knowledge of σ . Regarding the type of stability, $0 \in \mathbb{R}^n$ is said to be *globally uniformly asymptotically stable* (GUAS) under (1) when there is a $\mathcal{KL}$ -function β such that $\|x(k)\| \leq \beta(\|x_0\|, k)$ for all $k \in \mathbb{N}_{\geq 0}$, $x_0 \in \mathbb{R}^n$ and any switching signal σ , that is, the stability is uniform over all switching sequences.

To elucidate how even switched *linear* systems significantly differ from standard linear systems we mention the generalization of the spectral radius. Suppose that $\mathcal{F}_{\mathcal{A}}$ is the set of linear maps parametrized by the matrices $\mathcal{A} := \{A_1, \dots, A_{|\Sigma|}\}$ (i.e., $x(k+1) = A_{\sigma(k)}x(k)$). Stability of a switched system comprised by $\mathcal{F}_{\mathcal{A}}$ (or simply $\mathcal{A}$), under arbitrary switching, is captured by the *joint spectral radius* (JSR) (Rota and Strang, 1960) of $\mathcal{A}$, defined through

$$\rho(\mathcal{A}) := \lim_{k \rightarrow +\infty} \sup_{\{j_k, \dots, j_1\} \in \Sigma^k} \|A_{j_k} A_{j_{k-1}} \cdots A_{j_1}\|^{1/k},$$

not being larger than 1. In particular, 0 is GUAS under $\mathcal{F}_{\mathcal{A}}$ if and only if there is a $\gamma \in [0, 1)$ such that $\rho(\gamma^{-1}\mathcal{A}) \leq 1$. Unfortunately, computing $\rho(\mathcal{A})$ is NP-hard (Tsitsiklis and Blondel, 1997) and checking whether $\rho(\mathcal{A}) \leq 1$ is undecidable (Blondel and Tsitsiklis, 2000) and not algebraic (Kozyakin, 1990). Hence, one is drawn to the development of approximation schemes. We also remark that by exploiting the relationship between asymptotic and exponential stability, it can be shown that when 0 is GUAS under some $\mathcal{F}_{\mathcal{A}}$, then, a finite path-dependent quadratic Lyapunov always exists (Bliman and Ferrari-Trecate, 2003; Lee and Dullerud, 2006), however, the length of the path is not known *a priori*. This is the main roadblock, as a common quadratic Lyapunov need not exist, it is not clear, *a priori*, what to optimize over; not even for linear systems.

Then, to overcome some of these challenges, multiple Lyapunov functions can be employed (Branicky, 1998), that is, if at some instance a Lyapunov function V_a is decreasing along a state trajectory and we switch to some mode i , there should be some Lyapunov function V_b such that the decrease prevails (e.g., $V_a(x) \geq V_b(f_i(x))$). Unfortunately, there is no principled approach to select multiple Lyapunov functions, that is, one needs to select a *template* of Lyapunov functions $\mathcal{V}$, plus, a set of S Lyapunov functions $\{V_s \in \mathcal{V} \mid s \in S\}$ such that an appropriate decrease condition can be asserted. Towards addressing this, we are guided by the observation that inequalities of the form $V_a(x) \geq V_b(f_i(x))$ are captured by a directed edge $a \xrightarrow{i} b$. So, we can study stability through directed graphs representing these inequalities. This approach allows for unifying and generalizing the study of multiple Lyapunov functions; this is the framework of *path-complete Lyapunov functions* (PCLFs) (Ahmadi et al., 2014).

1.2 Path-complete Lyapunov functions

A labelled and directed graph $\mathcal{G} = (S, E)$ is *path-complete*, when for any (switching) sequence $\sigma(i_1)\sigma(i_2)\dots\sigma(i_k)$

[★] This work was supported by the ERC under the European Union's Horizon 2020 Research and Innovation Programme through Grant 864017 (L2C), it was announced in Jongeneel and Jungers (2025b).

there is a sequence of labelled edges $\{(s_j, s_{j+1}, \sigma(i_j))\}_{j=1,\dots,k}$ such that $(s_j, s_{j+1}, \sigma(i_j)) \in E$ for all $j \in [k]$.

Definition 1. (Path-complete Lyapunov function (PCLF)). For a switched system (1), parametrized by an alphabet Σ and a set of maps $\mathcal{F}$, let $\mathcal{V}$ be a template of candidate Lyapunov functions. Then, a PCLF for $\mathcal{F}$ is the pair $(\mathcal{G} = (S, E), V_S)$ where $\mathcal{G}$ is path-complete on Σ and $V_S = \{V_s \in \mathcal{V} \mid s \in S\}$ is a set of multiple Lyapunov functions such that for all $(a, b, i) \in E$ we have

$$V_a(x) \geq V_b(f_i(x)) \quad \forall x \in \mathbb{R}^n. \quad (2)$$

If this holds true, we denote this by $(\mathcal{G}, V_S) \in \text{pclf}(\mathcal{F}; \mathcal{V})$.

We emphasize that Definition 1 is common due to the focus on switched *linear* systems. One may extend the definition by requiring that there is a $\gamma \in \Gamma \subseteq \mathbb{R}_{\geq 0}$ such that for all $(a, b, i) \in E$ we have $V_a(x) \geq \gamma V_b(f_i(x))$ for all $x \in \mathbb{R}^n$, see Jongeneel and Jungers (2025a). Moreover, to clarify terminology. When $(\mathcal{G}, V_S) \in \text{pclf}(\mathcal{F}; \mathcal{V})$ we say that $\mathcal{G}$ is admissible for $(\mathcal{F}, \mathcal{V})$; and V_S is admissible for $(\mathcal{G}, \mathcal{F})$. To give an example, it follows that 0 is GUAS under the switched linear system comprised by $\mathcal{A} = \{A_1, \dots, A_{|\Sigma|}\}$ when there is a $\gamma \in [0, 1)$ such that there is PCLF for $\gamma^{-1}\mathcal{A}$ (Ahmadi et al., 2014, Thm. 2.4).

Now, towards a principled study of multiple Lyapunov functions, we need to be able to *compare* PCLFs.

Definition 2. (PCLF preorder). Consider two path-complete graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ on the alphabet Σ , a class of templates of Lyapunov functions $\mathcal{V}$ and a class of switching systems (i.e., subsets of maps) $\mathcal{F}$. Now, if for any $\mathcal{F} = \{f_i \mid i \in \Sigma\} \in \mathcal{F}$ and any $\mathcal{V} \in \mathcal{V}$, $\mathcal{G}_1$ being admissible for $(\mathcal{F}, \mathcal{V}) \implies \mathcal{G}_2$ being admissible for $(\mathcal{F}, \mathcal{V})$, then

$$\mathcal{G}_1 \leq_{\mathcal{V} \in \mathcal{V}, \mathcal{F} \in \mathcal{F}} \mathcal{G}_2. \quad (3)$$

One can interpret (3) as $\mathcal{G}_2$ being “*more expressive*” than $\mathcal{G}_1$ and in that sense $\mathcal{G}_2$ is understood as being “*better*”. To simplify notation, we might write $\mathcal{G}_1 \leq_{\mathcal{V}, \mathcal{F}} \mathcal{G}_2$.

We also recall that if (3) must hold for any class of maps $\mathcal{F}$ we write $\mathcal{G}_1 \leq_{\mathcal{V} \in \mathcal{V}} \mathcal{G}_2$ (or simpler $\mathcal{G}_1 \leq_{\mathcal{V}} \mathcal{G}_2$). If additionally, the relation is true for any class of templates $\mathcal{V}$ of candidate Lyapunov functions, we write $\mathcal{G}_1 \leq \mathcal{G}_2$. It is imperative to stress that we do not fix the dimension n here, see that graphs are path-complete, irrespective of the dimension of the underlying switched system.

Definition 2 readily implies the following. If we define the graph- and template-based approximation of $\rho(\mathcal{A})$ as

$$\rho_{\mathcal{G}, \mathcal{V}}(\mathcal{A}) := \inf_{\gamma > 0} \{\gamma : \exists (\mathcal{G}, V_S) \in \text{pclf}(\mathcal{F}_{\gamma^{-1}\mathcal{A}}; \mathcal{V})\},$$

we have that the JSR of $\mathcal{A}$ satisfies $\rho(\mathcal{A}) \leq \rho_{\mathcal{G}, \mathcal{V}}(\mathcal{A})$. In particular, if $\mathcal{F}$ comprises the set of linear maps and $\mathcal{G}_1 \leq_{\mathcal{V}, \mathcal{F}} \mathcal{G}_2$, then, $\rho_{\mathcal{G}_2, \mathcal{V}}(\mathcal{A}) \leq \rho_{\mathcal{G}_1, \mathcal{V}}(\mathcal{A})$ for any $\mathcal{A}$ (Debauche, 2024, Prop. 6.22).

Ideally, the ordering also helps in identifying procedures to *refine* graphs, that is, to find “*better*” graphs. With the previous discussion in mind, we would like to find maps L such that (at least) $\mathcal{G} \leq_{\mathcal{V}, \mathcal{F}} L(\mathcal{G})$. Of course, we are only genuinely refining $\mathcal{G}$ when $L(\mathcal{G}) \not\leq_{\mathcal{V}, \mathcal{F}} \mathcal{G}$. This desire, plus the desire to get a better grip on (3), led to a line of work on so-called “*lifting*” methods, which we briefly highlight below and with more details in Section 2.

To do so, we need the notion of a *simulation relation*.

Definition 3. (Simulation relation). The directed graph $\mathcal{G}_1 = (S_1, E_1)$ is said to simulate $\mathcal{G}_2 = (S_2, E_2)$ when there is a map $R : S_2 \rightarrow S_1$ such that for all $(a, b, i) \in E_2$ we have that $(R(a), R(b), i) \in E_1$.

Intuitively, Definition 3 is saying that $\mathcal{G}_1$ simulates $\mathcal{G}_2$ if the “*behaviour*” of $\mathcal{G}_2$ is contained in $\mathcal{G}_1$. Note that R need not be a bijection. A reason of interest is as follows. Suppose that $\mathcal{G}_1$ and $\mathcal{G}_2$ are path-complete on Σ and that $\mathcal{G}_1$ simulates $\mathcal{G}_2$. Let V_{S_1} be a set of admissible Lyapunov functions for $\mathcal{G}_1$, then, using the map $R : S_2 \rightarrow S_1$ from the simulation relation, we directly obtain admissible Lyapunov functions W_{S_2} for $\mathcal{G}_2$ by setting $W_s := V_{R(s)}$ for any $s \in S_2$, since for any $(a, b, i) \in E_2$ we have that

$$W_a(x) = V_{R(a)}(x) \geq V_{R(b)}(f_i(x)) = W_b(f_i(x)).$$

Then, the study of (3) became combinatorial (enabling tractability) as it was shown that $\mathcal{G}_1$ simulating $\mathcal{G}_2$ is equivalent to $\mathcal{G}_1 \leq \mathcal{G}_2$ (Philippe and Jungers, 2019, Thm. 3.5). Unfortunately, in general, $\mathcal{G}_1 \leq \mathcal{G}_2$ is significantly stronger than $\mathcal{G}_1 \leq_{\mathcal{V}, \mathcal{F}} \mathcal{G}_2$. However, several works showed that sometimes we can still “*lift*” $\mathcal{G}_1$ and recover similarly combinatorial results, see Theorem 5 below.

1.3 Research questions and contribution

In this note, we are inspired by the open problems posed in (Ahmadi et al., 2014, Sec. 7). Specifically, we ask: “*Given any pair $(\mathcal{V}, \mathcal{F})$, does there always exist a lift that turns the preorder $\leq_{\mathcal{V}, \mathcal{F}}$ combinatorial?*”

The contribution of this work is in providing the following. By means of Section 2 we illustrate the need for new and more generally applicable lifting techniques and we confirm with Theorem 9 that such a lift always exists, plus, with Theorem 16 we characterize in which sense we can even speak of *the* lift, given some preorder.

2. LIFTING TECHNIQUES

Below, we highlight some known lifting techniques, we point the reader to (Debauche, 2024, Ch. 7).

2.1 The sum lift

Suppose we have the following Lyapunov inequalities $V_a(x) \geq V_c(f_i(x))$ and $V_b(x) \geq V_d(f_i(x))$, then, if we define $x \mapsto V_{\{a,b\}}(x) := V_a(x) + V_b(x)$ and $x \mapsto V_{\{c,d\}}(x) := V_c(x) + V_d(x)$ we get that $V_{\{a,b\}}(x) \geq V_{\{c,d\}}(f_i(x))$. This is the main idea behind the *sum lift*.

Let $\text{mset}^T(S)$ be the *multiset*, of length T , of elements of S (e.g., $\text{mset}^2(\{a, b\}) = \{\{a, a\}, \{a, b\}, \{b, b\}\}$).

Definition 4. (The sum lift). Let $\mathcal{G} = (S, E)$ be a labelled and directed graph on the alphabet Σ , then, for any $T \in \mathbb{N}_{>0}$ define the T -sum lift of $\mathcal{G}$, denoted $\mathcal{G}^{\oplus T} = (S^{\oplus T}, E^{\oplus T})$, through the nodes $S^{\oplus T} = \text{mset}^T(S)$ and edges $E^{\oplus T} = \{((a_1, \dots, a_T), (b_1, \dots, b_T), i) \mid (a_k, b_k, i) \in E \forall k \in [T]\}$. The *sum lift* of $\mathcal{G}$ becomes $\mathcal{G}^{\oplus} = \bigcup_{T \in \mathbb{N}_{>0}} \mathcal{G}^{\oplus T}$.

The sum lift is particularly interesting due to the following result (Debauche et al., 2023, Thm. 1).

Theorem 5. (Sum lift simulation relation). Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be path-complete graphs, then, the following statements are equivalent:

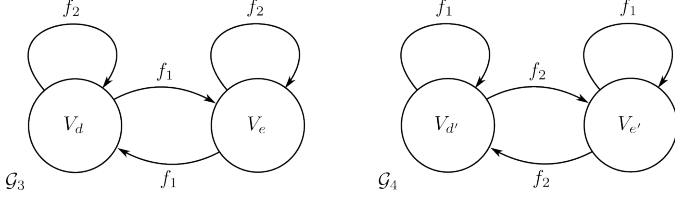


Fig. 1. Two path-complete graphs that cannot be simulated through a T -composition lift, yet, they can be related via a T -sum lift.

- (i) $\mathcal{G}_1^\oplus$ simulates $\mathcal{G}_2$;
- (ii) $\mathcal{G}_1 \leq_{\mathcal{V}} \mathcal{G}_2$ for any template $\mathcal{V}$ closed under addition.

One might hope that something like Theorem 5 holds when we restrict to the most well-studied setting closed under addition: *quadratic forms* and invertible linear maps. By means of a counterexample we recently showed that this is not the case (Jongeneel and Jungers, 2025a, Ex. II.1). Hence, we are motivated to study other preorders and lifts.

2.2 The composition lift

Given a Lyapunov inequality $V_a(x) \geq V_b(f_i(x))$, one can always pre-compose with f_j (i.e. $V_a(f_j(x)) \geq V_b(f_i(f_j(x)))$), so that if we define $V_{(a,j)} := V_a \circ f_j$ and $V_{(b,i)} := V_b \circ f_i$ we have $V_{(a,j)}(x) \geq V_{(b,i)}(f_j(x))$; a new Lyapunov inequality through *composition*.

Definition 6. (Composition lifts). Let $\mathcal{G} = (S, E)$ be a labelled and directed graph on the alphabet Σ . For any $T \in \mathbb{N}_{>0}$, the T -forward composition lift of $\mathcal{G}$, denoted $\mathcal{G}^{\circ T}$, is defined as follows. The nodes become $S^{\circ T} := \{(s, i_1, \dots, i_T) \mid s \in S, i_k \in \Sigma, k = 1, \dots, T\}$ and the edges are defined as follows: for any $(a, b, i) \in E$ and any T -tuple $(j_1, \dots, j_T) \in \Sigma^T$, $((a, j_1, \dots, j_T), (b, i, j_1, \dots, j_{T-1}), j_T) \in E^{\circ T}$. Now, let $\mathcal{G}^{\circ 0} := \mathcal{G}$, then, the forward composition lift of $\mathcal{G}$ is $\mathcal{G}^\circ = \bigcup_{T \in \mathbb{N}} \mathcal{G}^{\circ T}$.

Despite attractive properties of the T -composition lift (e.g., see Debauche (2024) and Jongeneel and Jungers (2025a)), also this lift provably fails to suffice for the quadratic template, in general. Specifically, we can find graphs that can be ordered, yet, there is no composition lift that leads to a simulation relation.

Example 7. (Lack of composition lift simulation relation). Consider the path-complete graphs $\mathcal{G}_3$ and $\mathcal{G}_4$ as given in Figure 1. It readily follows that both graphs are simply representing a common Lyapunov function under the assumption that we work with a template closed under *addition*. Indeed, the graphs can be compared directly through the T -sum lift. However, they cannot be compared via any T -composition lift. To see this, we do one direction explicitly, the rest follows similarly. Suppose that there is a $T \in \mathbb{N}$ such that $\mathcal{G}_3^{\circ T}$ simulates $\mathcal{G}_4$. Thus, there is a map $R : S_4 \rightarrow S_3^{\circ T}$ such that for all $(a, b, i) \in E_4$ we have $(R(a), R(b), i) \in E_3^{\circ T}$. Pick $(d', d', 1) \in E_4$. Then, since edges of $\mathcal{G}_3^{\circ T}$ are of the form $((a, j_1, \dots, j_T), (b, i, j_1, \dots, j_{T-1}), j_T)$ we see that for $(R(d'), R(d'), 1) \in E_3^{\circ T}$ to be true, we must have $(a, b, 1) \in E_3$ for some $a = b$. This cannot be true; a contradiction.

Although we have recently shown that we do not have the exact analogue of Theorem 5 for the composition

lift Jongeneel and Jungers (2025a), we infer the same limitation either way: the lift is not suitable for quadratic forms and linear invertible maps. This raises the question, does a suitable lift exist?

3. THE EXISTENCE OF GENERAL LIFTS

In Section 2 we saw that both the *sum lift* and the *composition lift* are not completely satisfactory when we restrict ourselves to the quadratic template. Then, could there be another lift that *is* satisfactory?

Recall that if $\mathcal{G}_1$ simulates $\mathcal{G}_2$, then $\mathcal{G}_1 \leq \mathcal{G}_2$. In particular, such a simulation relation implies that $\mathcal{G}_1 \leq_{\mathcal{V}, \mathcal{F}} \mathcal{G}_2$ for any $\mathcal{V}$ and $\mathcal{F}$. Now, let $\text{pc}(\Sigma) \subset \text{dgraph}(\Sigma)$ be the set of path-complete graphs (i.e., a subset of directed graphs) with labels taking values in some alphabet Σ . Then, in general, given a pair $(\mathcal{V}, \mathcal{F})$ one might aspire to find a lift $L : \text{pc}(\Sigma) \rightarrow \text{pc}(\Sigma)$ such that for any $\mathcal{G}_1, \mathcal{G}_2 \in \text{pc}(\Sigma)$ the following are equivalent: (i) $L(\mathcal{G}_1)$ simulates $\mathcal{G}_2$; and (ii) $\mathcal{G}_1 \leq_{\mathcal{V}, \mathcal{F}} \mathcal{G}_2$. However, one could go one step further and demand that for *any* preorder $\leq_p$ on $\text{pc}(\Sigma)$, implied by $\leq$ (n.b. we assume this from now on), there is some lift $L_p : \text{pc}(\Sigma) \rightarrow \text{pc}(\Sigma)$ such that the following are equivalent

- (A) $L_p(\mathcal{G}_1)$ simulates $\mathcal{G}_2$; and
- (B) $\mathcal{G}_1 \leq_p \mathcal{G}_2$.

Note that (A) and (B) together imply that the lift satisfies $\mathcal{G} \leq_p L_p(\mathcal{G})$ for $L_p(\mathcal{G})$ simulates $L_p(\mathcal{G})$. Thus, as by the equivalence of (A) and (B), $\mathcal{G}_1 \leq_p \mathcal{G}_2$ implies $L_p(\mathcal{G}_1) \leq_p \mathcal{G}_2$ and in particular

$$\mathcal{G}_1 \leq_p \mathcal{G}_2 \implies L_p(\mathcal{G}_1) \leq_p L_p(\mathcal{G}_2). \quad (4)$$

Recall that $\leq_p$ is a *preorder* on $\text{pc}(\Sigma)$, i.e., we have $\mathcal{G} \leq_p \mathcal{G}$ (*reflexivity*) and $\mathcal{G}_1 \leq_p \mathcal{G}_2$ together with $\mathcal{G}_2 \leq_p \mathcal{G}_3$ implies that $\mathcal{G}_1 \leq_p \mathcal{G}_3$ (*transitivity*). Thus, (4) is equivalent to saying that L_p must be continuous (C^0) with respect to the *Alexandroff topology* on $\text{pc}(\Sigma)$ induced by the preorder $\leq_p$ (see Theorem 19). That is, we have the following.

Lemma 8. (Lifts are continuous). For any lift $L_p : \text{pc}(\Sigma) \rightarrow \text{pc}(\Sigma)$ such that (A) and (B) are equivalent, we have that L_p is continuous with respect to the Alexandroff topology induced by the preorder $\leq_p$.

In the (induced by $\leq_p$) Alexandroff topology, (*basic*) open sets are of the form

$$U_{\mathcal{G}} := \{\mathcal{G}' \in \text{pc}(\Sigma) \mid \mathcal{G} \leq_p \mathcal{G}'\}. \quad (5)$$

For example, open sets in the Alexandroff topology induced by $\leq_{\mathcal{V} \in \mathcal{V}, \mathcal{F}}$ are simply of the form $\bigcap_{\mathcal{V} \in \mathcal{V}} \{\mathcal{G}' \in \text{pc}(\Sigma) \mid \mathcal{G} \leq_{\mathcal{V}, \mathcal{F}} \mathcal{G}'\}$.

Now, recall Definition 2, then, all of the above leads us to the following existence result. The intuition being that if (A) and (B) should be equivalent, then $L_p(\mathcal{G})$ must “contain all that is better than $\mathcal{G}$ ”.

Theorem 9. (Existence of general lifts). For any preorder $\leq_p$ on $\text{pc}(\Sigma)$, there exists a map $L_p : \text{pc}(\Sigma) \rightarrow \text{pc}(\Sigma)$, continuous with respect to the Alexandroff topology induced by $\leq_p$, such that (A) and (B) are equivalent.

Proof. We can define the map $L_p : \text{pc}(\Sigma) \rightarrow \text{pc}(\Sigma)$ by

$$\mathcal{G} \mapsto L_p(\mathcal{G}) := \bigcup_{\mathcal{G}' \in U_{\mathcal{G}}} \mathcal{G}'. \quad (6)$$

It readily follows that L_p is continuous with respect to the Alexandroff topology induced by $\leq_p$ (again, by the equivalence with monotonicity as in (4), see Theorem 19). Also, note that by the construction of L_p , $L_p(\mathcal{G})$ is path-complete whenever $\mathcal{G}$ is (*i.e.*, as it is the disjoint union of path-complete graphs). Now for the implications.

(A) $\implies$ (B): Since $\leq_p$ is implied by $\leq$, this follows directly from

$$\mathcal{G}_1 \stackrel{(6)}{\leq_p} L_p(\mathcal{G}_1) \stackrel{(A)}{\leq} \mathcal{G}_2.$$

(B) $\implies$ (A): Since $\mathcal{G}_1 \leq_p \mathcal{G}_2$ we know that $\mathcal{G}_2 \in U_{\mathcal{G}_1}$. Thus, $\mathcal{G}_2$ is an (isolated) subgraph of $L_p(\mathcal{G}_1)$, thereby, the simulation relation holds trivially. $\square$

Theorem 9 implies in particular that the preorder (3) can *always* be turned into a combinatorial statement, opening the door for computations. Here, we are motivated by tractable results for the sum lift, *e.g.*, see Philippe et al. (2018) and (Debauche et al., 2024, Sec. 7.3.5).

3.1 Topological properties of lifts

With some abuse of notation, we let $\mathcal{G}_0$ be the canonical common Lyapunov function. Then, since $\mathcal{G}_0 \leq_p \mathcal{G}$ for all $\mathcal{G} \in \text{pc}(\Sigma)$, we have that there exists a continuous path $\gamma : [0, 1] \rightarrow \text{pc}(\Sigma)$ with $\gamma(0) = \mathcal{G}_0$ and $\gamma(1) = \mathcal{G}$ (see Section A.2). This implies the following proposition.

Proposition 10. ($\text{pc}(\Sigma)$ is contractible). Given any preorder $\leq_p$, then, the space $\text{pc}(\Sigma)$ endowed with the Alexandroff topology induced by $\leq_p$ is contractible.

Proof. We need to show that the identity map on $\text{pc}(\Sigma)$ is homotopic to a constant map. To that end, consider the map C_0 defined through $\text{pc}(\Sigma) \ni \mathcal{G} \mapsto C_0(\mathcal{G}) := \mathcal{G}_0$. Regarding the homotopy, consider the map $H : [0, 1] \times \text{pc}(\Sigma) \rightarrow \text{pc}(\Sigma)$ defined by $H(0, \mathcal{G}) := \mathcal{G}_0$ and $H(t, \mathcal{G}) := \mathcal{G} \forall t \in (0, 1]$. Note that both C_0 and $\text{id}(\cdot; \text{pc}(\Sigma))$ are continuous on $\text{pc}(\Sigma)$. Now, pick any open set $U \subseteq \text{pc}(\Sigma)$ and suppose that $\mathcal{G}_0 \notin U$, then, $H^{-1}(U) = (0, 1] \times U$. On the other hand, if $\mathcal{G}_0 \in U$, then since $\mathcal{G}_0 \leq_p \mathcal{G}$ for all $\mathcal{G} \in \text{pc}(\Sigma)$, we have that $U = \text{pc}(\Sigma)$ and thus the preimage under H is simply $[0, 1] \times \text{pc}(\Sigma)$. Either way, the preimage of U is open and hence H is continuous. $\square$

It is of importance to interpret contractibility in the light of underlying topology, that is, in the light of (5).

In general, contractibility relates to the existence of global sections and a form of minimality (Farber, 2003, Thm. 1).

Denote by $\tau(\leq')$ the Alexandroff topology induced by a preorder $\leq'$ and let $\tau(\text{co.o.})$ denote the *compact-open topology*, *e.g.*, see Fox (1945).

Lemma 11. (Homotopy equivalence). Given any preorder $\leq_p$, then, for any lift L_p such that (A) and (B) are equivalent, L_p is homotopic to the identity map on $\text{pc}(\Sigma)$ with respect to the Alexandroff topology induced by $\leq_p$.

Proof. Define the map $H : [0, 1] \times \text{pc}(\Sigma) \rightarrow \text{pc}(\Sigma)$ through $H(0, \mathcal{G}) := \mathcal{G}$ and $H(t, \mathcal{G}) := L_p(\mathcal{G}) \forall t \in (0, 1]$. By Lemma 8, both $\text{id}(\cdot; \text{pc}(\Sigma))$ and L are continuous on $\text{pc}(\Sigma)$. Now, pick any open set $U \subseteq \text{pc}(\Sigma)$ and suppose that $\mathcal{G} \in U$. Then, $(0, \mathcal{G}) \in H^{-1}(U)$. However, since $\mathcal{G} \in U$,

$U_{\mathcal{G}} \subseteq U$ and thus $L_p(\mathcal{G}) \in U$, as $\mathcal{G} \leq_p L(\mathcal{G})$. Therefore, $(t, \mathcal{G}) \in H^{-1}(U) \forall t \in [0, 1]$. As $\mathcal{G}$ was arbitrary, this means that $\{[0, 1] \times U\} \cup \{(0, 1] \times L_p^{-1}(U)\} = H^{-1}(U)$, which is open in the product topology. Indeed, the preimage can be further simplified to $[0, 1] \times U$ by noting that $L(\mathcal{G}) \leq_p \mathcal{G}$ (*i.e.*, for $V := L_p^{-1}(U)$ we have $V \subseteq U$).

By transitivity, Proposition 10 and Lemma 11 imply that L_p is homotopic to a constant map, on $(\text{pc}(\Sigma), \tau(\leq_p))$.

Next, we start the comparison of lifts and topologies. The following result follows directly from our standing assumptions (*i.e.*, $\mathcal{G}_1 \leq \mathcal{G}_2 \implies \mathcal{G}_1 \leq_p \mathcal{G}_2$).

Lemma 12. (Comparing induced topologies). The identity map from $(\text{pc}(\Sigma), \tau(\leq))$ to $(\text{pc}(\Sigma), \tau(\leq_p))$ is continuous.

Proposition 13. (On the topology of lifts). Let $\mathcal{L}(\Sigma)$ be the space of continuous functions L from $(\text{pc}(\Sigma), \tau(\leq))$ to itself such that there exists a preorder $\leq_p$ for which L is the corresponding lift such that (A) and (B) are equivalent. Then,

- (i) any lift L_p that is such that (A) and (B) are equivalent, for some preorder $\leq_p$, is contained in $\mathcal{L}(\Sigma)$; and
- (ii) the space $(\mathcal{L}(\Sigma), \tau(\text{co.o.}))$ is contractible.

Proof. (i): First, we show that any lift L_p that is such that (A) and (B) are equivalent, is continuous on $(\text{pc}(\Sigma), \tau(\leq))$. To that end, note that for any preorder $\leq_p$ we have $\mathcal{G}_1 \leq \mathcal{G}_2 \implies \mathcal{G}_1 \leq_p \mathcal{G}_2$, but then, since $\mathcal{G}_2 \leq_p L(\mathcal{G}_2)$ (*e.g.*, since $L(\mathcal{G}_2)$ trivially simulates $L(\mathcal{G}_2)$), we have that $\mathcal{G}_1 \leq_p L(\mathcal{G}_2)$, which is equivalent to $L(\mathcal{G}_1) \leq L(\mathcal{G}_2)$ by the equivalence of (A) and (B). Therefore, any lift is continuous with respect to $\tau(\leq)$. Note, this also shows that L is continuous as a map from $(\text{pc}(\Sigma), \tau(\leq_p))$ to $(\text{pc}(\Sigma), \tau(\leq))$.

(ii): Now, following the same line of arguments as in Lemma 11, we have that $L \simeq_h \text{id}(\cdot; \text{pc}(\Sigma))$ on $(\text{pc}(\Sigma), \tau(\leq))$. Note, here we exploit $L(\mathcal{G}) \leq \mathcal{G}$ and use a homotopy of the form $H(0, \mathcal{G}) := L(\mathcal{G})$ and $H(t, \mathcal{G}) := \mathcal{G}$ for all $t \in (0, 1]$. Specifically, as L is continuous, it follows that for any open set $U \in \tau(\leq)$ we have that $H^{-1}(U) = \{[0, 1] \times L^{-1}(U)\} \cup \{(0, 1] \times U\}$, which is open. Note that $\text{id}(\cdot; \text{pc}(\Sigma))$ is also a valid lift (for the preorder $\leq$).

Then, to pass from homotopies to continuous paths in functions spaces under $\tau(\text{co.o.})$, we note that Alexandroff spaces are first countable (see Appendix A), such that we can appeal to (Fox, 1945, Thm. 2) and conclude that $\mathcal{L}(\Sigma)$ is path-connected. One can also directly work with the function space as shown next.

Regarding contractibility, in what follows, $\text{pc}(\Sigma)$ is understood to be topologized by $\tau(\leq)$. Then, recall that all sets of the form $\mathcal{O}(K, U) := \{\ell \in C^0(\text{pc}(\Sigma); \text{pc}(\Sigma)) \mid \ell(K) \subseteq U\}$, with $K \subseteq \text{pc}(\Sigma)$ compact and $U \subseteq \text{pc}(\Sigma)$ open, comprise a subbasis for $\tau(\text{co.o.})$ on $\mathcal{L}(\Sigma)$. Now consider the map $\mathcal{H} : [0, 1] \times \mathcal{L}(\Sigma) \rightarrow \mathcal{L}(\Sigma)$ defined through $\mathcal{H}(0, L) := L$ and $\mathcal{H}(s, L) := \text{id}(\cdot; \text{pc}(\Sigma))$ for all $s \in (0, 1]$. Continuity of $\mathcal{H}$ follows when for any $\mathcal{O}(K, U)$ as above $\mathcal{H}^{-1}(\mathcal{O}(K, U))$ is open. Note that for any $L \in \mathcal{O}(K, U)$ we have that $\text{id}(\cdot; \text{pc}(\Sigma)) \in \mathcal{O}(K, U)$ since if $L(\mathcal{G}) \in U$, then $\mathcal{G} \in U$ by $L(\mathcal{G}) \leq \mathcal{G}$. As such, $\mathcal{H}^{-1}(\mathcal{O}(K, U)) = \{[0, 1] \times \mathcal{O}(K, U)\} \cup \{(0, 1] \times \mathcal{L}(\Sigma)\}$. $\square$

We remark that the contractability results from above all relate to having minimal (e.g., $\mathcal{G}_0 \leq_p \mathcal{G}$) or maximal (e.g., $L \leq \text{id}(\cdot; \text{pc}(\Sigma))$) elements.

3.2 Quotient spaces

Given two graphs $\mathcal{G}_1$ and $\mathcal{G}_2$, one might have a preorder $\leq_p$ such that $\mathcal{G}_1 \leq_p \mathcal{G}_2$, but this does not prevent $\mathcal{G}_2 \leq_p \mathcal{G}_1$ from also being true, even if $\mathcal{G}_1$ and $\mathcal{G}_2$ are not isomorphic.

To overcome this form of redundancy, one can pass to a quotient space. For any preorder $\leq_p$, we can readily extract a *partially ordered set* (poset) by defining the equivalence $\sim_p$ through $\mathcal{G}_1 \sim_p \mathcal{G}_2$ if $\mathcal{G}_1 \leq_p \mathcal{G}_2$ and $\mathcal{G}_2 \leq_p \mathcal{G}_1$, that is, $(\text{pc}(\Sigma)/\sim_p, \leq_p)$ is now a poset. Let $[\mathcal{G}]_p := \{\mathcal{G} \in \text{pc}(\Sigma) \mid \mathcal{G} \sim_p \mathcal{G}'\}$ denote the corresponding equivalence classes. As a poset is in particular a preorder and the quotient map $[L_p]_p : \text{pc}(\Sigma)/\sim_p \rightarrow \text{pc}(\Sigma)/\sim_p$ preserves the order, this map is again a continuous map in the sense of Alexandroff. One readily obtains the following result.

Lemma 14. (The quotient lift is the identity). For any lift L_p such that (A) and (B) are equivalent, for some preorder $\leq_p$, we have that the quotient map $[L_p]_p : \text{pc}(\Sigma)/\sim_p \rightarrow \text{pc}(\Sigma)/\sim_p$ is the identity on $\text{pc}(\Sigma)/\sim_p$.

Proof. Consider (A) and (B). See that $\mathcal{G} \leq_p \mathcal{G}$ is trivially true and thus $L_p(\mathcal{G})$ simulates $\mathcal{G}$, but then $L_p(\mathcal{G}) \leq \mathcal{G}$ and thus $L_p(\mathcal{G}) \leq_p \mathcal{G}$. On the other hand, $L_p(\mathcal{G})$ evidently simulates $L_p(\mathcal{G})$ such that $\mathcal{G} \leq_p L_p(\mathcal{G})$ must hold. This means that $L_p(\mathcal{G}) \sim_p \mathcal{G}$, that is, for all $[\mathcal{G}]_p \in \text{pc}(\Sigma)/\sim_p$ we have $[L_p]_p([\mathcal{G}]_p) = [\mathcal{G}]_p$. $\square$

We also point out that as the quotient map $q_p : \text{pc}(\Sigma) \rightarrow \text{pc}(\Sigma)/\sim_p$ is a homotopy equivalence (McCord, 1966, Thm. 4), $\text{pc}(\Sigma)/\sim_p$ is contractible by Proposition 10.

Regarding the approximation of the JSR, one readily obtains the following from Lemma 14.

Corollary 15. (Lifts cannot refine the JSR). Let $\mathcal{F}$ comprise all subsets of linear maps $\{f_i \mid i \in \Sigma\}$, then, for any template $\mathcal{V}$ and lift L on $\text{pc}(\Sigma)$ such that (A) and (B) are equivalent, for the preorder $\leq_{\mathcal{V}, \mathcal{F} \in \mathcal{F}}$, we have that

$$\rho_{\mathcal{G}, \mathcal{V}}(\mathcal{F}_A) = \rho_{L(\mathcal{G}), \mathcal{V}}(\mathcal{F}_A), \quad \forall \mathcal{F}_A \in \mathcal{F}.$$

At last, we formalize in which sense we can speak of *the* lift L_p (i.e., we characterize equivalence of valid lifts).

Theorem 16. (Equivalence of lifts). Given a preorder $\leq_p$, let $\mathcal{L}_p(\Sigma)$ be the space of lifts L_p such that (A) and (B) are equivalent. Then,

- (i) all $L_p \in \mathcal{L}_p(\Sigma)$ are continuous from $(\text{pc}(\Sigma), \tau(\leq_p))$ to $(\text{pc}(\Sigma), \tau(\leq))$;
- (ii) for any $L_p^\alpha, L_p^\beta \in \mathcal{L}_p(\Sigma)$ we have that $L_p^\alpha(\mathcal{G}) \sim L_p^\beta(\mathcal{G})$ $\forall \mathcal{G} \in \text{pc}(\Sigma)$; and
- (iii) the space $(\mathcal{L}_p(\Sigma), \tau(\text{co.o.}))$ is contractible.

Proof. (i): This follows directly by the same argument as in the proof of Proposition 13.

(ii): Suppose that L_p^α and L_p^β are such that (A) and (B) are equivalent for $\leq_p$. Therefore, both $\mathcal{G} \leq_p L_p^\alpha(\mathcal{G})$ and $\mathcal{G} \leq_p L_p^\beta(\mathcal{G})$ hold, but then by (A) $\implies$ (B) $L_p^\alpha(\mathcal{G}) \leq L_p^\beta(\mathcal{G})$ and $L_p^\beta(\mathcal{G}) \leq L_p^\alpha(\mathcal{G})$ for any $\mathcal{G} \in \text{pc}(\Sigma)$.

(iii): By (i) and (ii), we can fix some $L_p \in \mathcal{L}_p(\Sigma)$ and know that for any other $L'_p \in \mathcal{L}_p(\Sigma)$ we have that $L'_p(\mathcal{G}) \leq L_p(\mathcal{G})$ for all $\mathcal{G} \in \text{pc}(\Sigma)$. However, then this map L_p takes precisely the role of $\text{id}(\cdot; \text{pc}(\Sigma))$ as in Proposition 13 (change the topology of the domain to $\tau(\leq_p)$). $\square$

3.3 Comparing preorders

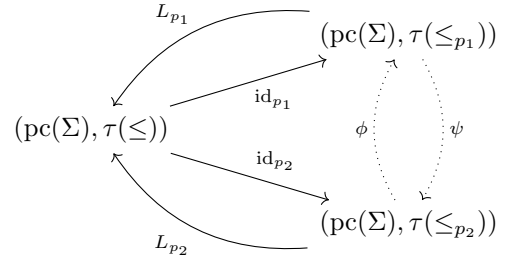
An underexplored property of these lifts is that they map graphs from $(\text{pc}(\Sigma), \tau(\leq_p))$ to $(\text{pc}(\Sigma), \tau(\leq))$. This allows for comparing preorders.

For instance, let L_p be a lift corresponding to $\leq_p$ and let $L_{p'}$ be a lift corresponding to $\leq_{p'}$. Then, if $\mathcal{G}_1 \leq_p \mathcal{G}_2$ and $L_{p'}(\mathcal{G}_1) \leq L_p(\mathcal{G}_1)$ we have that $\mathcal{G}_1 \leq_{p'} \mathcal{G}_2$.

Proposition 17. (Quasi-order embedding). Let $\leq_{p_1}$ and $\leq_{p_2}$ be preorders, then there exist continuous maps ϕ and ψ such that for any $\mathcal{G}_1, \mathcal{G}_2 \in \text{pc}(\Sigma)$ we have

$$\begin{aligned} \mathcal{G}_1 \leq_{p_1} \mathcal{G}_2 &\implies \psi(\mathcal{G}_1) \leq_{p_2} \psi(\mathcal{G}_2); \text{ and} \\ \mathcal{G}_1 \leq_{p_2} \mathcal{G}_2 &\implies \phi(\mathcal{G}_1) \leq_{p_1} \phi(\mathcal{G}_2). \end{aligned}$$

Proof. Using Lemma 12 and Proposition 13, then, the following diagram



can be completed by $\phi := \text{id}_{p_1} \circ L_{p_2}$ and $\psi := \text{id}_{p_2} \circ L_{p_1}$. $\square$

4. CONCLUDING REMARKS

We have shown that lifts, in the context of PCLFs, always exist. A natural follow-up question is to capture when these lifts have a finite representation. Differently put, how to enrich the structural demands (A) and (B)?

Appendix A. THE ALEXANDROFF TOPOLOGY

Typically, topologies are constructed such that arbitrary unions of open sets are open, yet, only *finite* intersections of open sets are again open, with the example to have in mind being $\bigcap_{n=1}^{\infty} (-1/n, 1/n) = \{0\}$. An Alexandroff topology is different in that *arbitrary* intersections of open sets are now declared to be open as well, see Alexandroff (1937). As this topology is not well-known, we provide some key results below. For more, we refer to May (2016).

A.1 The relation to preorders

In what follows we work using the convention that sets of the form (5) are declared open (e.g. $U_x = \{x' \in X \mid x \leq x'\}$), we note that some authors flip the preorder and subset inclusion to connect Alexandroff spaces and preorders cf. (May, 2016, Rem. 1.6.5).

Let (X, τ_A) be a topological space with τ_A the Alexandroff topology. As such, we can define the following open sets $U_x = \bigcap \{U \in \tau_A \mid x \in U\}$. It readily follows that the

relation $\leq_A$ defined through $x \leq_A y$ if $U_y \subseteq U_x$ renders $\langle X, \leq_A \rangle$ a preorder, mostly due to transitivity of subset inclusion. Note that $\{U_x\}_{x \in X}$ is a (minimal) basis for (X, τ_A) . In fact, note that Alexandroff spaces X are *first-countable* as $\{U_x\}$ functions as a (countable) neighbourhood basis for $x \in X$. Now suppose we have a preorder $\langle Y, \leq \rangle$ and define the sets $U_y := \{y' \in Y \mid y \leq y'\}$. Then, $\{U_y\}_{y \in Y}$ is a basis for a topology on Y since $y \in U_y$ and if $y \in U_{y_1} \cap U_{y_2}$ (i.e., $y_1 \leq y$ and $y_2 \leq y$), then $U_y \subseteq (U_{y_1} \cap U_{y_2})$. Now to see that the topology is Alexandroff, consider the intersection of open sets $U := \bigcap_{i \in I} U_{y_i}$, we can assume that $U \neq \emptyset$ such that there is some $y \in U$. This implies that $U_y \subseteq U_{y_i}$ for all $i \in I$, but that means that U is the union of all these open sets U_y . We recover the following.

Theorem 18. There is a one-to-one correspondence between the Alexandroff topologies τ_A and the preorders $\leq$ on a set X .

As the Alexandroff topology is intimately related to order, so are continuous maps with respect to this topology. For simplicity we do not emphasize that the topologies and preorders on X and Y may be different.

Let X and Y be preorders endowed with an Alexandroff topology. Now, suppose that $f : X \rightarrow Y$ is continuous and $x_1 \leq x_2$. Since f is continuous, $f^{-1}(U_{f(x_1)})$ must be open, in particular, since $x_1 \in U_{x_1}$, $f(x_1) \in U_{f(x_1)}$ and $x_2 \in U_{x_1}$ it follows that $x_2 \in U_{x_1} \subseteq f^{-1}(U_{f(x_1)})$. But then, $f(x_2) \in U_{f(x_1)}$ and thus $f(x_1) \leq f(x_2)$. On the other hand, suppose that $f : X \rightarrow Y$ preserves order, that is, if $x_1 \leq x_2$, then $f(x_1) \leq f(x_2)$. Let V be open in Y and say that $f(x_1) \in V$, this implies that $U_{f(x_1)} \subseteq V$. But then, since f preserves order, if $x_1 \leq x_2$, $f(x_2) \in U_{f(x_1)} \subseteq V$ and thus $x_2 \in f^{-1}(V)$. So, since x_2 was arbitrary, $U_{x_1} \subseteq f^{-1}(V)$ and thus, since x_1 was any point such that $f(x_1) \in V$, $f^{-1}(V)$ will be the union of all these U_{x_1} that comply. Thus, we have the following.

Theorem 19. Let X and Y be preorders endowed with the Alexandroff topology, then $f : X \rightarrow Y$ is continuous if and only if f preserves the preorder.

A.2 Continuous paths

Let $x_1 \leq x_2$, then, there is continuous path $\gamma : [0, 1] \rightarrow X$ such that $\gamma(0) = x_1$ and $\gamma(1) = x_2$. One can simply set $\gamma(0) = x_1$ and $\gamma(t) = x_2$ for all $t \in (0, 1]$. The reason being that for any open set $U \subseteq X$ we have either: (i) $x_1, x_2 \notin U$ and so $\gamma^{-1}(U) = \emptyset$; (ii) $x_2 \in U$ but $x_1 \notin U$ such that $\gamma^{-1}(U) = (0, 1]$; or (iii) $x_1 \in U$, but then since $x_1 \leq x_2$, x_2 must be in U as well (by the construction of the open sets) and thus $\gamma^{-1}(U) = [0, 1]$.

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