

On capturing linear controllability through a Conley index^{*}

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Abstract: The Conley index of an isolated invariant set is a topological invariant that captures qualitative dynamical behaviour in a neighbourhood of this set. In this note we show that Conley indices can capture controllability of linear control systems, both in the continuous-time and discrete-time case. In particular, by means of an appropriately designed smooth feedback, a single Conley index can capture linear controllability.

Keywords: controllability, linear control systems, topological index theory

1. INTRODUCTION

We are witnessing renewed interest in qualitative (topological) dynamical systems theory in part due to its application in fields where precise models are scarce, like synthetic biology (Cummins et al., 2017; Gameiro et al., 2024).

Here, one frequently appeals to *Conley's index theory* (Conley, 1978). To introduce this index, consider Figure 1 (bottom right). Suppose one is given the set N with the knowledge that some flow leaves N through the exit set L , then, the homotopy type of N/L is a signature of the qualitative behaviour corresponding to an unstable hyperbolic equilibrium point with a 2-dimensional unstable manifold. In fact, as $N/L \neq \{*\}$, we know that N must contain some isolated invariant set S ; the homotopy type of $(N/L, [L])$ is the Conley index of S . See Section 2.3 for details.

Conley's theory allows one to draw qualitative conclusions based on coarse information. For example, although the direct study of periodic orbits is hard, one might be able to infer that a particular genetic regulatory network is able to generate a Conley index corresponding to a periodic orbit (McCord et al., 1995). Indeed, the great generality and wide applicability of Conley's theory makes the framework attractive, e.g., one can study PDEs (Rybakowski and Zehnder, 1985). Towards even wider applicability and rigorous numerics, we are currently witnessing its "combinatorization" (Kalies et al., 2014, 2016; Mrozek, 2017; Kalies et al., 2022; Lipiński et al., 2023). Moreover, data-driven inference is developed, e.g., see Yim and Nanda (2025). One can also study coarse models *globally* by computing what is called a *Morse-Conley graph*, e.g., see Vieira et al. (2022). At last, we mention that the Conley index

has been used to study stability of dynamical systems (Hui and Haddad, 2006; Moulay and Hui, 2011), however, to the best of our knowledge, never to infer control theoretic properties.

Nonetheless, from a control perspective, it seems natural to ask if these indices can be of use beyond the study of *fixed* dynamical systems. In particular, one might ask:

"Given a control system and the ability to generate a set of Conley indices, e.g., by employing several feedback controllers, what does this imply regarding the underlying control system?"

For such a program to be of any value, it must work out in the simplest case. Therefore, in this note we start this study and address the question:

"Is there a topological index that captures controllability of linear control systems?"

To be more precise, we address the question for finite-dimensional, deterministic, time-invariant linear control systems and show that the Conley index indeed captures controllability, e.g., see Section 3. Moreover, we show that there is a smooth feedback such that the Conley index corresponding to the closed-loop system is a signature of controllability, that is, a single index, generated under smooth feedback, indicates if a linear control system is controllable, see Theorem 3.7. We believe that this result encourages further study of control theoretic counterparts of Conley's theory, see also Colonius and Kliemann (2012).

We already emphasize that the condition we will derive is inherently trajectory-based and therefore amendable to data-driven implementations. Compared to known data-driven techniques, e.g., see (van Waarde et al., 2025, §3.2), we also point out that the input signal we employ is a smooth static feedback and not a time-varying signal.

In the next section we provide some background material on Conley index theory, in Section 3 we detail our main results, in Section 4 we provide a few examples and in Section 5 we conclude.

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2. TECHNICAL PRELIMINARIES

In this section we briefly highlight linear controllability and the definition of the Conley index. To improve accessibility, we provide some of the topological background.

2.1 Linear controllability

Although this work is motivated by nonlinear phenomena, we consider linear time-invariant (LTI) continuous-time (CT) and discrete-time (DT) deterministic control systems of the form

$$\dot{x}(t) = Ax(t) + Bu(t), \quad t \in \mathbb{R}_{\geq 0}, \quad (1)$$

and

$$x_{k+1} = Ax_k + Bu_k, \quad k \in \mathbb{Z}_{\geq 0}, \quad (2)$$

where $x \in \mathbb{R}^n$ is the state variable and $u \in \mathbb{R}^m$ is the input. When we select for u some function of x , e.g., $x \mapsto \mu(x)$, we speak of a *static state feedback* and the closed-loop system becomes $\dot{x} = Ax + B\mu(x)$. Note, we usually omit the explicit dependence on time. The feedback is linear when there is a matrix $K \in \mathbb{R}^{m \times n}$ such that $\mu(x) = Kx$.

The control system (1) (a similar statement holds for (2)) is said to be **controllable** when for any two points $x_a, x_b \in \mathbb{R}^n$ there exists a finite $T > 0$ and an integrable input signal $[0, T] \ni t \mapsto \nu(t)$ such that the resulting solution of (1) is such that $x(0) = x_a$ and $x(T) = x_b$.

Then, both (1) and (2) are controllable if and only if

$$(B \ AB \ \dots \ A^{n-1}B) \quad (3)$$

is full-rank, e.g., see (Sontag, 1998, Thm. 2-3). This is Kalman's celebrated rank condition and allows for speaking of the *pair* (A, B) being controllable. Nonlinear generalizations of this rank condition exist, e.g., see (Nijmeijer and van der Schaft, 1990, Ch. 3).

2.2 Homotopy types

We will use the Conley index to capture controllability. The Conley index itself is in its most general form the *homotopy type* of a pointed topological space.

Two continuous maps $f, f' : (X, x_0) \rightarrow (Y, y_0)$, over the *pointed topological spaces* (X_0, x_0) and (Y_0, y_0) , are *homotopic, relative to x_0* , denoted $f \simeq_h f' \text{ rel } x_0$, when $f(x_0) = y_0$, $f'(x_0) = y_0$ and there is a continuous map $H : X \times [0, 1] \rightarrow Y$ such that $H(\cdot, 0) = f$, $H(\cdot, 1) = f'$ and $H(x_0, s) = y_0$ for all $s \in [0, 1]$. Then, (X, x_0) and (Y, y_0) are of the same **homotopy type** when there are maps $f : (X, x_0) \rightarrow (Y, y_0)$ and $g : (Y, y_0) \rightarrow (X, x_0)$ such that $f \circ g \simeq_h \text{id}_Y \text{ rel } y_0$ and $g \circ f \simeq_h \text{id}_X \text{ rel } x_0$. Let $[X, x_0]$ denote the *pointed homotopy type* of a pointed space (X, x_0) . In particular, let Σ^k denote $[\mathbb{S}^k, s_0]$ for s_0 some point on $\mathbb{S}^k$, that is, Σ^k is the homotopy type of the pointed k -sphere. To interpret $\Sigma^0 = [\mathbb{S}^0, s_0]$, we recall that $\mathbb{S}^0 \cong \{0, 1\}$ (we will use “ $\cong$ ” to denote appropriate isomorphisms). At last, the wedge sum of two pointed spaces (X_0, x_0) and (Y_0, y_0) is $X \vee Y := X \sqcup Y / \sim$, where $\sim$ denotes the identification of x_0 and y_0 . For reference, see (Hatcher, 2002, Ch. 0).

2.3 The Conley index

A particularly fruitful topological framework to study qualitative dynamics is Conley's generalization of Morse's

index theory (Conley, 1978). We briefly highlight his *index theory*, first in the context of flows.

Continuous-time Given a flow φ on a locally compact metric space M , i.e., a continuous map $\varphi : M \times \mathbb{R} \rightarrow M$ such that $\varphi^0 = \text{id}_M$ and $\varphi^{s+t} = \varphi^s \circ \varphi^t$, for $\varphi^t(\cdot) := \varphi(\cdot, t)$, then, $S \subseteq M$ is said to be an *isolated invariant set* when there is a compact set $K \subseteq M$, called an *isolating neighbourhood*, such that $S = \text{inv}(K, \varphi) := \{p \in K \mid \varphi(p, \mathbb{R}) \subset K\} \subseteq \text{int}(K)$. Note, S may be empty. If no confusion can arise, we drop the dependence on φ . Now, let $\text{cl}(X)$ denote the closure of X , then, a pair of compact sets (N, L) , with $L \subseteq N$ is an **index pair** for S when

- (i) $S = \text{inv}(\text{cl}(N \setminus L))$ and $N \setminus L$ is a neighbourhood of S ;
- (ii) L is positively invariant in N , i.e., if $x \in L$ and $\varphi^{[0, t]}(x) \in N$ for some $t > 0$, then $\varphi^{[0, t]}(x) \in L$; and
- (iii) L is an exit set for N , i.e., if for $x \in N$ there is a $s > 0$ such that $\varphi^s(x) \notin N$, then there is a $\tau \in (0, s)$ such that $\varphi^\tau(x) \in L$.

Then, the (CT) homotopy **Conley index** of S (under φ) is the homotopy type of the pointed space $(N/L, [L])$, denoted by $h(S, \varphi)$. Again, if no confusion can arise, we drop the dependence on φ .

We recall that $N/\emptyset := N \sqcup \{*\}$. Moreover, the Conley index of an isolated invariant set S always exists and is independent of the choice of index pair, e.g., see (Mischaikow and Mrozek, 2002, §3.1). Also, if an isolated invariant set splits as $S = S_1 \cup S_2$, with $S_1 \cap S_2 = \emptyset$, then $h(S) = h(S_1) \vee h(S_2)$ (Srzednicki, 2004, Thm. 9.3 (ii)), plus, see that $[N/L]$ is invariant under a change of coordinates, e.g., consider $N/L = N \setminus L \cup [L]$ under some homeomorphism.

Then, to do computations (e.g., see Kaczynski et al. (2004)), a homological variant has been developed, that is, one computes $H_*(N/L, [L])$. Under mild assumptions¹ on the pair (N, L) , one can exploit that $H_*(N/L, [L]) \cong H_*(N, L)$. In fact, one can always find some index pair such that $H_*(N/L, [L]) \cong H_*(N, L)$ holds true (Mischaikow and Mrozek, 2002, Rem. 3.9). What is more, we may assume that N and L are CW complexes, e.g., see Manolescu (2013).

Example 2.1 (Conley indices). In Figure 1 we show 4 examples of 2-dimensional flows, depicting the trivial Conley index ($S = \emptyset$) and the pointed k -sphere for $k \in \{0, 1, 2\}$. Indeed, typically one does not know S , one rather works with some isolating neighbourhood N . An important result, the Ważewski Property, is that if $h(\text{inv}(N))$ is not trivial, then $\text{inv}(N) \neq \emptyset$, e.g., see (Srzednicki, 2004, Thm. 9.3 (i)). We also mention that not just any invariant set comes with an index pair (i.e., when the set is not *isolated*). For instance, suppose that the flow φ is a linear center on $\mathbb{R}^{2n}$, then there cannot be a compact neighbourhood W of 0 such that $0 = \text{inv}(W, \varphi)$, as W always contains some small ball of radius $r > 0$ such that $B_r(0) := \{x \in \mathbb{R}^{2n} \mid \|x\|_2 < r\} \subseteq \text{inv}(W, \varphi)$.

¹ Using standard notation, suppose that (X, A) is a *good pair* (Hatcher, 2002, p. 114), then $H_*(X, A) \cong \tilde{H}_*(X/A)$, but then $(X/A, [A])$ is also a good pair and thus $H_*(X/A, [A]) \cong H_*(X, A)$.

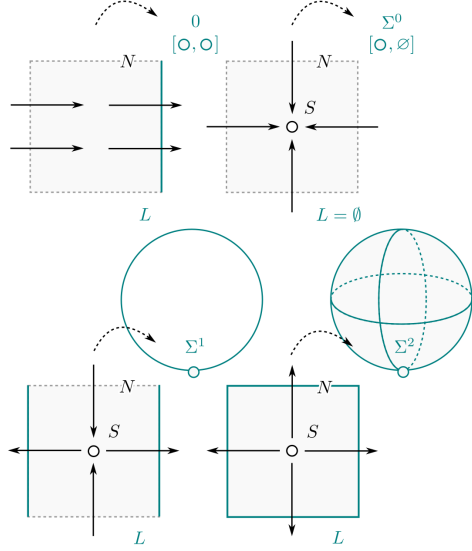


Fig. 1. Several Conley indices, for flows, from trivial (top left) to the pointed k -spheres $\{\Sigma^0, \Sigma^1, \Sigma^2\}$.

Looking at (1), one should observe that if we want to use the Conley index to infer information about the control system, we ideally input a feedback for u , and not a time-varying signal, this, such that the resulting closed-loop system naturally provides us with a flow.

Discrete-time The Conley index for discrete-time dynamical systems (maps) has been largely propelled by Robbin and Salamon (1988). The intuition is the same as before, but now, the homotopy type of the space $(N/L, [L])$ is not invariant under a choice of index pairs², the reason being that previously, homotopies could be constructed via the flows, this is not the case with maps. We will need different machinery. There are several different approaches, e.g., Franks and Richeson (2000) define an index via filtration pairs and shift equivalence. This construction is general, but not very transparent, however. For recent developments see Weilandt (2019).

Instead, we follow the homological approach as taken by Mrozek (1990). Note already that if we want $\{0\} \subset \mathbb{R}^n$ to be some isolated set under a continuous linear map $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$, f must be a homeomorphism. Hence, without loss of generality, we will work with homeomorphisms.

Following (Mischaikow and Mrozek, 2002, Def. 3.24), given a homeomorphism $f : M \rightarrow M$, a pair of compact sets (N, L) , with $L \subseteq N$ is an index pair for an isolated invariant S when

- (i) $\text{cl}(N \setminus L)$ is an isolating neighbourhood of S ;
- (ii) $f(L) \cap N \subseteq L$; and
- (iii) $f(N \setminus L) \subseteq N$.

We have selected this definition due to its clear connection with the continuous-time definition cf. (Mrozek, 1990, Def. 2.1). As mentioned before, the homotopy type of N/L does not carry the right information, instead consider the following map.

Given some index pair (N, L) , let $q : N \rightarrow N/L$ be the corresponding quotient map, i.e., $q : x \mapsto [x]$ and define $f_{N,L} : N/L \rightarrow N/L$ through

$$[x] \mapsto f_{N,L}([x]) := \begin{cases} [f(x)] & \text{if } f(x) \in N \\ [L] & \text{otherwise} \end{cases}, \quad (4)$$

which can be shown to be continuous, e.g., see (Franks and Richeson, 2000, Thm. 3.8). We need the following notion to utilize these maps.

Let X and Y be topological spaces, then, two continuous maps $f : X \rightarrow X$ and $g : Y \rightarrow Y$ are **shift equivalent**, denoted $f \simeq_s g$, when there are maps $r : X \rightarrow Y$ and $s : Y \rightarrow X$ such that the following diagrams commute

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ r \downarrow & & \downarrow r \\ Y & \xrightarrow{g} & Y \end{array} \quad \begin{array}{ccc} Y & \xrightarrow{g} & Y \\ s \downarrow & & \downarrow s \\ X & \xrightarrow{f} & X \end{array},$$

plus, there is a $m \in \mathbb{Z}_{\geq 0}$ such that $r \circ s = g^m$ and $s \circ r = f^m$. Shift equivalence can be understood as a weaker version of *topological equivalence*, but note that if f and g are homeomorphisms, then they are shift equivalent if and only if they are topologically equivalent, i.e., $m = 0$, $h \circ f = g \circ h$ for $h = r$ and $h^{-1} = s$. This means that shift equivalence captures more than stability, orientation of the map also plays a role, see (Robinson, 1995, Ch. 4) for the linear case.

Then, it can be shown that if (N, L) and (N', L') are both index pairs for S , under f , then $f_{N,L} \simeq_s f_{N',L'}$ (Franks and Richeson, 2000, Thm. 4.3). Franks and Richeson (2000) use this observation to define a homotopy index directly, which is, however, difficult to work with as one needs to study an equivalence class of maps. Instead, we use the link with Mrozek (1990) as discussed in (Mischaikow and Mrozek, 2002, §3.2).

Let $\text{gker}(e) := \bigcup_{n \in \mathbb{N}_{\geq 0}} \ker(e^n)$ be the generalized kernel of some endomorphism $e : E \rightarrow E$, of a vector spaces E . Let

$$\bar{e} : E/\text{gker}(e) \ni [x] \rightarrow [e(x)] \in E/\text{gker}(e),$$

then, $\bar{e}$ restricted to $\text{gim}(\bar{e}) := \bigcap_{n \in \mathbb{N}_{\geq 0}} \text{im}(\bar{e}^n)$, is called the *Leray reduction* of e , see (Mrozek, 1990, §4). The Leray reduction is used since one can show that e is shift equivalent to its Leray reduction (Mischaikow and Mrozek, 2002, Prop. 3.30). To work with vector spaces, we appeal to homology with rational coefficients.

Now, given an index pair (N, L) such that $H_*(N/L, [L]; \mathbb{Q})$ is finite-dimensional, we can define the (DT) homological **Conley index**, for S under a map³ f , denoted $\text{Con}_*(S, f)$, as the pair

- (i) $CH_*(S, f) := H_*(N/L, [L]; \mathbb{Q})/\text{gker}(f_{N,L})_*$;
- (ii) $\chi_*(S, f) : CH_*(S, f) \rightarrow CH_*(S, f)$

with the latter being the Leray reduction of $(f_{N,L})_*$.

For us, the following is important, see (Mrozek, 1990, Thm. 3.1) and (Mischaikow and Mrozek, 2002, Thm. 3.33). Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a linear hyperbolic map⁴, with

² See (Mischaikow and Mrozek, 2002, §3.2) for some examples.

³ Usually restricted to compact spaces, although what we need is to be able to pick a compact index pair (N, L) .

⁴ An invertible linear map with no eigenvalues in $\{z \in \mathbb{C} \mid |z| = 1\}$.

a k -dimensional unstable invariant subspace and l real eigenvalues below -1 , then,

$$\text{Con}_i(\{0\}, f) = \begin{cases} 0 & i \neq k \\ (\mathbb{Q}, (-1)^l \text{id}) & i = k. \end{cases} \quad (5)$$

Given (5), we believe that Con_* is the right discrete index for our purposes. Indeed, when computing this index for a time-one map of a flow, one recovers the (cohomological) Conley index of the flow (Mrozek, 1990, Thm. 2.9). Moreover, this index captures the *orientation* of a map, something that one has control over under controllability assumptions. See Jongeneel and Kuhn (2021) for an example of orientation in the context of linear optimal control.

3. LINEAR CONTROLLABILITY AND CONLEY INDICES

We start with the continuous-time case.

Proposition 3.1 (A homotopy Conley index condition for controllability of linear CT control systems). *The linear control system (1) is controllable if and only if we can find a set of linear feedback maps $\{\mu_i\}_{i=0}^n$, that generate the set $\{\Sigma^i\}_{i=0}^n$ of pointed spheres, as the Conley index of $\{0\} \subset \mathbb{R}^n$.*

Proof. First, controllability of the pair (A, B) in the context of (1) is equivalent to arbitrary pole-placement (Wonham, 1979, Thm. 2.1), *e.g.*, for any $k \in \{0, 1, \dots, n\}$ we can always find a matrix $K_k \in \mathbb{R}^{m \times n}$ such that 0 is a hyperbolic equilibrium point under the closed-loop system matrix $A + BK_k$, with the first k eigenvalues of $A + BK_k$ being elements of the set $\{z \in \mathbb{C} \mid \Re(z) > 0\}$. Differently put, we can decompose the tangent space at 0 in a k -dimensional unstable manifold and an $(n - k)$ -dimensional stable manifold. As such, we can generate any pointed k -sphere Σ^k , for $k \in \{0, 1, \dots, n\}$, as the Conley index of 0 under some linear ODE $\dot{x} = (A + BK_k)x$. Explicitly, after a change of coordinates, we can consider the Jordan normal form of $A + BK_k$, we may even assume the matrix to be real and diagonal by selecting appropriate poles. Then, in these new coordinates we have $h(\text{inv}(\text{cl}(N_k/L_k))) = \Sigma^k$ for $N_k = [-1, 1]^n$ and L_k selected as the set of faces of ∂N_k that intersect with the unstable subspace under $A + BK_k$.

For the other direction, assume that for any $k \in \{0, 1, \dots, n\}$ we can find linear feedback $x \mapsto \mu_k(x) := K_k x$ such that we can generate any pointed k -sphere, as the Conley index of 0 under some linear ODE $\dot{x} = (A + BK_k)x$. Suppose, for the sake of contradiction, that we can do this while (A, B) is *not* a controllable pair. After a linear change of coordinates (*e.g.* $x \mapsto Tx$ for some $T \in \text{GL}(n, \mathbb{R})$), the control system (1) can always be put into its control canonical form

$$\begin{pmatrix} \dot{x}_c \\ \dot{x}_o \end{pmatrix} = \begin{pmatrix} A_c & \tilde{A} \\ 0 & A_o \end{pmatrix} \begin{pmatrix} x_c \\ x_o \end{pmatrix} + \begin{pmatrix} B_c \\ 0 \end{pmatrix} u, \quad (6)$$

with $(x_c, x_o) \in \mathbb{R}^{c+o} = \mathbb{R}^n$. This result is known as *Kalman's controllability decomposition* (Sontag, 1998, Lem. 3.3.3). Note that such a transformation is a diffeomorphism of the space that fixes the origin. As such, we have that the Conley index of 0, generated under μ_k , is preserved. Denote the new index pair by (N_T, L_T) . Then, however, if (A, B) is not controllable, the uncontrollable subspace (the o -dimensional subspace containing $(0, x_o)$)

is non-empty. This means there is no selection of input that can alter the dynamics $\dot{x}_o = A_o x_o$, contradicting that $(N_T/L_T, [L_T])$ can have the homotopy type of *any* pointed k -sphere Σ^k , see also (Mrozek, 1990, Thm. 3.13). $\square$

We readily obtain a rank condition, which seems interesting in the light of (3).

Corollary 3.2 (A homology Conley index condition for controllability of linear CT control systems). *The system (1) is controllable if and only if we can find a set of linear feedback maps $\{\mu_i\}_{i=0}^n$, resulting in index pairs $\{(N_i, L_i)\}_{i=0}^n$ for $\{0\} \subset \mathbb{R}^n$, such that*

$$\text{rank} \bigoplus_{i=0}^n H_i(N_i, L_i; \mathbb{Z}) = n + 1. \quad (7)$$

Remark 3.3 (On the rank). Kalman's controllability matrix (3) should be of rank n to assert controllability, clearly not $n + 1$. However, we do not study a parametrization of the system, but its trajectories. Condition (7) is expressing that one must be able to generate all $n + 1$ scenarios of instability, that is, all $n + 1$ "different exit sets".

Now see that the condition from Proposition 3.1 can be simplified, essentially since no uncontrollable subspace can prevail if we can generate both Σ^0 and Σ^n .

Corollary 3.4 (A simpler homotopy Conley index condition for controllability of linear CT control systems). *The linear control system (1) is controllable if and only if we can find a set of linear feedback maps $\{\mu_0, \mu_n\}$, that generate the set $\{\Sigma^0, \Sigma^n\}$ of pointed spheres, as the Conley index of $\{0\} \subset \mathbb{R}^n$.*

From Corollary 3.4 one naturally recovers the following. Note, *stabilizability* of (1) means that there exists a $K \in \mathbb{R}^{m \times n}$ such that $A + BK$ is Hurwitz.

Corollary 3.5 (A homotopy Conley index condition for stabilizability of linear CT control systems). *The linear control system (1) is stabilizable if and only if we can find a linear feedback map μ_0 that generates Σ^0 as the Conley index of $\{0\} \subset \mathbb{R}^n$.*

Next, the discrete-time analogue of Corollary 3.4 is as follows. Here, we are "generating" closed-loop maps. One may also generalize Proposition 3.1 as linear controllability implies we have control over the orientation of the map.

Corollary 3.6 (A homotopy Conley index condition for controllability of linear DT control systems). *The linear control system (2) is controllable if and only if we can find a set of linear feedback maps $\{\mu_0, \mu_n\}$ that generate $\{\text{Con}_0(\{0\}, \cdot) \cong (\mathbb{Q}, \cdot), \text{Con}_n(\{0\}, \cdot) \cong (\mathbb{Q}, \cdot)\}$, as degrees of the (DT) homological Conley index of $\{0\} \subset \mathbb{R}^n$.*

Proof. The proof is similar to that of Proposition 3.1 and Corollary 3.4 and readily follows from (5). $\square$

3.1 A single controllability index

Given Section 3, one might wonder if we can find a *single* smooth (C^∞) feedback and then a *single* index that capture controllability. By interpolating appropriately between the feedback terms from Corollary 3.4, we are able to show this is true. We only provide the CT argument.

Theorem 3.7 (A single homotopy Conley index condition for controllability of linear CT control systems). *The*

linear control system (1) is controllable if and only if we can find a smooth feedback map μ , that generates $\Sigma^n \vee \Sigma^1 \vee \Sigma^0$, as a Conley index for some invariant set $S \subset \mathbb{R}^n$ that partitions as follows:

- (i) $S := S_a \cup \{0\}$ for $S_a \cap \{0\} \neq \emptyset$ and $S_a \neq \emptyset$; and
- (ii) $h(S_a) = \Sigma^n \vee \Sigma^1$ and $h(\{0\}) = \Sigma^0$.

Proof. Suppose that (1) is controllable, then, there are feedback gains K_1 and K_2 such that $F_1 := (A + BK_1)$ and $F_2 := -(A + BK_2)$ are Hurwitz, i.e., 0 is globally exponentially stable (GES) under $\dot{x} = F_i x$, $i = 1, 2$. Note, for F_2 we have flipped the sign as we want instability through K_2 .

Then, the above means that we can find quadratic Lyapunov functions $V_i : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$, of the form $V_i(x) := \langle P_i x, x \rangle$, $P_i \succ 0$, asserting that 0 is GES under $\dot{x} = F_i x$ (Sontag, 1998, Thm. 18), i.e., $\langle \nabla V_i(x), F_i x \rangle < 0 \forall x \in \mathbb{R}^n \setminus \{0\}$. Now, define the ellipsoids $E_i(c_i) := \{x \in \mathbb{R}^n \mid V_i(x) \leq c_i\}$. Set $c_1 > 0$ and $c_2 := 2 \max_{x \in \partial E_1(c_1)} V_2(x)$. We have $E_1(c_1) \subset \text{int}(E_2(c_2))$. Then, we can find a smooth (C^∞) bump function $b_1 : \mathbb{R}^n \rightarrow [0, 1]$ such that $b_1|_{E_1(c_1)} = 1$ and $\text{supp}(b_1) \subseteq \text{int}(E_2(c_2))$ (Lee, 2012, Prop. 2.25). Similarly, we can find a smooth function b_2 that equals 1 on $\mathbb{R}^n \setminus \text{int}(E_2(c_2))$ and has its support in $\mathbb{R}^n \setminus E_1(c_1)$. Then, define the smooth feedback $\mu : \mathbb{R}^n \rightarrow \mathbb{R}^m$ through $\mu(x) := \sum_{i=1}^2 b_i(x) K_i x$. Indeed, we may assume that the resulting vector field $x \mapsto Ax + B\mu(x)$ is globally Lipschitz. We denote the corresponding flow by φ_μ .

Define $r := (1/2) \min_{x \in \partial E_1(c_1)} \|x\|_2$ and set $N_0 := B_r(0)$. Then, (N_0, L_0) , for $L_0 := \emptyset$, is an index pair for 0. Additionally, set $N_a := E_2(c_2) \setminus \text{int}(E_1(c_1))$, then (N_a, L_a) , for $L_a := \partial N_a$ is an index pair. As $h(\text{inv}(\text{cl}(N_a \setminus L_a)), \varphi_\mu) \cong \Sigma^n \vee \Sigma^1$ is non-trivial, the pair corresponds to some non-empty isolated invariant set. At last, the disjoint unions $N := N_0 \cup N_a$ and $L := L_a \cup L_0$ give rise to an index pair, for some isolated invariant set containing 0, that generates the desired Conley, e.g., see (Szrednicki, 2004, Thm. 9.3) for properties of the Conley index that were used.

Now, suppose that we can generate $\Sigma^n \vee \Sigma^1 \vee \Sigma^0$ as the Conley index of some isolated invariant S that contains 0 and satisfies Property (i) and (ii) from the theorem. For the sake of contradiction suppose that we can do this without (1) being controllable. Again, we can factor our control system as (6). However, then to be able to generate the $\mathbb{S}^0$ part of $h(S, \varphi_\mu)$, we must have that A_o is Hurwitz.

Now, given any index pair (N, L) and let $O \subseteq \mathbb{R}^n$ be the uncontrollable subspace, we know that $L \cap O = \emptyset$ since A_o is Hurwitz by assumption. Then, using the flow $\psi^t(x) := e^{tA_o} x$, we can strongly deformation retract $\mathbb{R}^n$ onto $\mathbb{R}^n \setminus O =: \mathbb{R}^c$, relative to L , where $c < n$ (e.g., reparametrize time). This means that the homotopy type of $(N/L, [L])$ is preserved throughout the deformation retract. However, then it follows that we cannot generate $\mathbb{S}^n$, as this is obstructed by the lower-dimensional space $\mathbb{R}^c$, e.g., for any compact CW pair (N', L') in $\mathbb{R}^c$, $H_{k>c}(N'/L'; \mathbb{Z}) \cong 0$ (Hatcher, 2002, Lem. 2.34 (b)), while $H_{n>c}(\mathbb{S}^n; \mathbb{Z}) \cong \mathbb{Z}$. Hence, a contradiction. $\square$

We have visualized Theorem 3.7 in Figure 2.

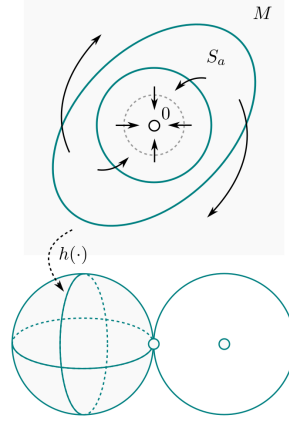


Fig. 2. Visualization of the construction in Theorem 3.7, e.g., here, the Conley index corresponding to S_a becomes $\Sigma^2 \vee \Sigma^1$.

4. EXAMPLES

First, we highlight that all of the above remains true under a change of coordinates.

Example 4.1 (A linear control system in disguise). For the following known example (Nijmeijer and van der Schaft, 1990, Ex. 5.4), as given by

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} x_2 - 2x_2x_3 + x_3^2 \\ x_3 \\ 0 \end{pmatrix} + u \begin{pmatrix} 4x_2x_3 \\ -2x_3 \\ 1 \end{pmatrix} \quad (8)$$

it is known that the nonlinear control system (8) is *strongly accessible* for all $x \in \mathbb{R}^3$ (e.g., see (Nijmeijer and van der Schaft, 1990, Ch. 3)) and moreover we can find a smooth change of coordinates, turning (8) into a controllable linear system (Nijmeijer and van der Schaft, 1990, Thm. 5.3). Thereby, we can find smooth feedback to generate Σ^k , with $k \in \{0, \dots, 3\}$, as the (global) Conley index of $\{0\} \subset \mathbb{R}^3$, under (8).

In line with Example 4.1, one may also extend some of our results to the case of *feedback linearization* of nonlinear input-affine control systems. In particular, the nonlinear generalization of (6) is enabled by Frobenius' theorem, e.g., see (Nijmeijer and van der Schaft, 1990, Thm. 3.49).

Next, we provide a concrete example of Theorem 3.7, one that can be visualized.

Example 4.2 (An integrator system). Consider the trivial 1-dimensional control system

$$\dot{x} = u.$$

Now, consider the feedback $\mu(x) := \sum_{i=1}^2 (-1)^i b_i(x)x$ for

$$b_2(x) := \begin{cases} 0 & 0 \leq |x| \leq (3/2)\pi \\ (1/2) \sin(|x|) + 1/2, & (3/2)\pi \leq |x| \leq (5/2)\pi \\ 1 & \text{otherwise} \end{cases}$$

and $b_1(x) := 1 - b_2(x)$. The resulting dynamical system is visualized in Figure 3.

5. CONCLUSION, NONLINEARITIES AND OTHER FUTURE WORK

Regarding generalizations to generic nonlinear control systems, consider the following example.

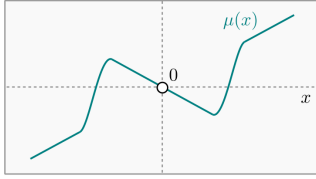


Fig. 3. The smooth feedback map μ from Example 4.2, resulting in the dynamical system $\dot{x} = \mu(x)$.

Example 5.1 (A bilinear control system). Consider the bilinear control system $\dot{x} = xu$. Clearly, this system is not controllable at $x = 0$, yet, we can, for instance, impose the smooth feedback $x \mapsto \mu_s(x) := -x^2$ and render 0 globally asymptotically stable (GAS). Similarly, we can use $x \mapsto \mu_u(x) := x^2$ to render 0 GAS, *backwards in time*. Differently put, for any $k \in \{0, 1\}$, we can find smooth feedback such that the Conley index of $\{0\} \subset \mathbb{R}$ is Σ^k .

Given the example and results from above, we believe it is worthwhile to further study the precise implications of being able to generate certain Conley indices. Can one develop a qualitative controllability theory? Moreover, what can be said if we include the structure of Morse-Conley graphs? We are particularly interested in the case of periodic orbits cf. McCord et al. (1995).

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