Asymptotic stability equals exponential stability—*while* you twist your eyes

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Abstract

Suppose that two vector fields on a smooth manifold render some equilibrium point globally asymptotically stable (GAS). We show that there exists a homotopy between the corresponding semiflows such that this point remains GAS along this homotopy.

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1 Introduction In the context of what we call today *Conley index theory* (see Section A), Charles C. Conley himself posed the following "*converse question*" in the late 1970s: "*To what extent does the homotopy index* [Conley index] *itself determine the equivalence class of isolated invariant sets which are related by continuation?*" [Con78, p. 83]. Then, recently, Matthew D. Kvalheim proved that uniquely integrable C^0 vector fields, on a C^∞ manifold M, rendering a compact set $A \subseteq M$ asymptotically stable, are homotopic on an open neighbourhood $U \supseteq A$ such that throughout the homotopy the vector fields do not vanish on $U \setminus A$ [Kva23, Thm. 1]. Connecting this result to Conley's question, a follow-up question (revitalized) by Kvalheim—which is the central question of this note—is the following:

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Question: "*Are, dynamical systems that render a set A asymptotically stable, homotopic through dynamical systems that preserve this notion of stability?*"

This question, in one form or another, inspired several works, for instance, [Rei92; MRS00; JS24]. Here, we will elaborate on commentary by the author in [JS24]. Specifically, in the seminal paper "*Asymptotic stability equals exponential stability, and ISS equals finite energy gain—if you twist your eyes*" from 1999, Lars Grüne, Eduardo D. Sontag and Fabian R. Wirth showed that asymptotic stability "*equals*" exponential stability in the sense that if an equilibrium point is asymptotically stable under some vector field on $\mathbb{R}^n$, then, there is a suitable change of coordinates rendering this point exponentially stable [GSW99]. Such a change of coordinates is understood to be *instantaneous*. However, by leveraging their work, we show in this note that asymptotic stability can (almost) always be *continuously* "transformed" into exponential stability, while preserving asymptotic stability throughout the transformation, see Theorem 3.4. Differently put, asymptotic stability equals exponential stability—not only *if* you twist your eyes, but *while* you twist your eyes. The key to this is to observe that we can in fact select the transformation of Grüne, Sontag and Wirth to be an element of the *orientation-preserving* homeomorphism *group* on $\mathbb{R}^n$, not just any homeomorphism *cf.* [JS24, Sec. III], see the proof of Proposition 3.2.

This result provides a partial solution to Conley's converse question as it turns out that the asymptotically stable systems under consideration can be continuously transformed into the same exponentially stable system and hence, by transitivity, into each other. Concurrently, we emphasize in this note that results of this form typically extend to discontinuous vector fields when solutions are understood *in the sense of Filippov*. We also discuss intimate connections with optimization (*e.g.*, see Example 3.2) and optimal transport (*e.g.*, see Example 4.1).

1.1 Related work It can be argued that questions of the form above emerged from studies aimed at classifying manifolds, maps, vector fields and so forth. A successful, yet coarse, resolution has been found in the study of these objects *up to homotopy*, *e.g.*, motivated early on by the fundamental group being homotopy invariant, Hopf's degree theorem, CW complexes, intractability of topological equivalence and more work at the intersection of topology and dynamical systems.

We cannot do justice here to the wealth of work in this area, but let us mention that inspired by Aleksandr Andronov, Lev Pontryagin, René Thom, Maurico Peixoto, George D. Birkhoff, John Milnor and several others, it was in particular Stephen Smale highlighting that this intersection is an interesting one for *both* topology and dynamical systems. As put by Sheldon Newhouse: "*... one of the great influences that Steve had, at least on me and I think many others, aside from excitement, was the whole idea that one might get a structure theory for general dynamical systems. That such a general structure theory for most systems could exist I think was a goal of Poincare and Birkhoff, and somehow in the intervening years, it was lost sight of by people working on specialized problems. The idea of getting a global structure of all systems has spawned a lot of development in the past, and it is still going on.*" [HMS12, p. 183].

Indeed, after seminal work by Smale on the qualitative classification of dynamical systems [Sma67], several homotopy results appeared being concerned with *Morse-Smale vector fields*, *e.g.*, see [Asi75; NP76; Fra79]. Similarly, one can study if gradient vector fields are homotopic through gradient vector fields, *e.g.*, while restricting equilibria [Par90], see also [Rei91; Kva23] for work close in spirit to ours. These works were typically concerned with structural stability and the study of the topology of spaces of dynamical systems, that is, to understand when dynamical systems are in some sense close or equivalent, *e.g.*, see [SS75] for pointers and further results.

Then, in control theory, these tools (*i.e.*, certain homotopy invariants) were used to construct necessary conditions for feedback controllers to exist, *e.g.*, if a desirable (closed-loop) dynamical system belongs to a certain homotopy class, then, there must be at least exist *some* feedback that renders the control system a member of this class, regardless of it resulting in the actual desirable system, *e.g.*, see [Bro83; KZ84; Zab89; Cor90].

Most of these results have in common that the original objects are homotoped to something simple, something "*canonical*" where we do our analysis and computations. For instance, in the context of control systems, when our goal is stabilization of the origin on $\mathbb{R}^n$ through feedback, then, the canonical differential equation is $\dot{x} = -x$ and we would like our closed-loop dynamical system to be in some qualitative sense equivalent to this equation. This is precisely how the "*index condition*" by Krasnosel'skiĭ and Zabreĭko is derived [KZ84, Sec. 52]. Clearly, necessary conditions of this form are only as valuable as the homotopy class is distinctive, *e.g.*, although the index condition is powerful, it cannot differentiate between $\dot{x} = x$ and $\dot{x} = -x$ on $\mathbb{R}^2$. This is precisely what motivates us, we expect that understanding Conley's converse question can lead to stronger necessary conditions for continuous stabilizing feedback to exist.

Besides possibly stronger necessary conditions for continuous stabilization, the study of Conley's converse question directly relates to understanding topological properties of spaces of stable dynamical systems (as was partial motivation for our previous work [JS24]). This is of practical importance as several frameworks in optimal control and reinforcement learning aim to optimize over precisely such a space, *e.g.*, moving from a pre-stabilized system to an optimally controlled system, we point to [Ber+17; Faz+18; Wan+22; FGFT24] and references therein.

Before detailing further technicalities, we recall that Conley's converse question is not trivial.

Example 1.1 (Trivial convex combinations can fail)**.** Consider a linear differential equation $\dot{x} = A(s)x$ on $\mathbb{R}^2$ parametrized by the matrices

$$[0,1] \ni s \mapsto A(s) := s \cdot \begin{pmatrix} -1 & 10 \\ 0 & -1 \end{pmatrix} + (1-s) \cdot \begin{pmatrix} -1 & 0 \\ 10 & -1 \end{pmatrix}.$$

Both $A(0)$ and $A(1)$ correspond to global asymptotically stable systems, yet, for $s = (1/2)$ we find that the system $\dot{x} = A(s)x$ is unstable. Hence, we cannot just construct straight-line homotopies between stable vector fields and expect that stability is preserved. We know from [JS24] that instead, for linear vector fields we should homotope via the canonical ODE $\dot{x} = -x$. Explicitly, one could consider the following path of linear vector fields defined by

$$H(x;s) := \begin{cases} \begin{pmatrix} -1 & (1-2s)\cdot 10 \\ 0 & -1 \end{pmatrix} x & s \in [0, 1/2] \\ \begin{pmatrix} -1 & 0 \\ (2s-1)\cdot 10 & -1 \end{pmatrix} x & s \in (1/2, 1] \end{cases}.$$

$\circ$

Notation and preliminaries Let $r \in \mathbb{N} \cup \{\infty\}$, then, $C^r(U;V)$ denotes the set of C^r-smooth functions from U to V. The inner product on $\mathbb{R}^n$ is denoted by $\langle \cdot, \cdot \rangle$ and $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : \|x\|_2 = 1\}$ is the embedded unit sphere in $\mathbb{R}^n$. By $\mathrm{cl}(W)$ we denote the (topological) closure of a set $W \subseteq M$ and by $\mathrm{int}(W)$ we denote its interior. The identity map $p \mapsto p$ on a space M is denoted by id_M and tangent spaces of sufficiently regular manifolds N are denoted by T_qN, for $q \in N$, with TN denoting the corresponding tangent bundle. Given a function $V : M \to \mathbb{R}$, let $V^{-1}(c)$ denote the level set $\{p \in M : V(p) = c\}$. We typically use φ to denote a (semi)flow. If this flow comes from a vector field X, we write $\varphi(\cdot; X)$, similarly, if φ is parametrized by $s \in [0,1]$ we write $\varphi(\cdot; s)$, that is, we typically overload the meaning of φ.

A homotopy $[0,1]\times M \ni (s,p)\mapsto H(s,p)$ is said to be an *isotopy* when $p\mapsto H(s,p)$ is a topological embedding for all $s\in[0,1]$. A homeomorphism $\psi:\mathbb{R}^n\to\mathbb{R}^n$ is said to be a *stable homeomorphism* when it is a finite composition of homeomorphisms that equal the identity map on some—not necessarily the same—non-empty open subset of $\mathbb{R}^n$. We denote by $\mathrm{Homeo}(\mathbb{R}^n;\mathbb{R}^n)$ the group (under composition) of homeomorphisms $\psi:\mathbb{R}^n\to\mathbb{R}^n$. Additionally, we denote by $\mathrm{Homeo}^+(\mathbb{R}^n;\mathbb{R}^n)\subset\mathrm{Homeo}(\mathbb{R}^n;\mathbb{R}^n)$ the subgroup of all *orientation-preserving*[1] homeomorphisms.

A function $\alpha:\mathbb{R}_{\geq 0}\to\mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{K}$ when α is strictly increasing and $\alpha(0)=0$. If, additionally $\lim_{s\to+\infty}\alpha(s)=+\infty$, then, α is of class $\mathcal{K}_\infty$. These type of functions are called *comparison functions*, *e.g.*, see [Hah67; Kel14].

1.2 Dynamical systems In this note we study deterministic, finite-dimensional, time-invariant, continuous, (global) *semi-dynamical systems* comprised of the triple $\Sigma:=(M^n,\varphi,\mathbb{R}_{\geq 0})$. Here, M^n will be a smooth n-dimensional manifold diffeomorphic to $\mathbb{R}^n$, which we denote by $M^n\simeq_d\mathbb{R}^n$, and $\varphi:\mathbb{R}_{\geq 0}\times M^n\to M^n$ is a global *semiflow*, that is, a continuous map that satisfies for any $p\in M^n$: (*i*) $\varphi(0,p)=p$ (the *identity* axiom); and (*ii*) $(\varphi(s,\varphi(t,p))=\varphi(t+s,p)\ \forall s,t\in\mathbb{R}_{\geq 0}:=\{t\in\mathbb{R}:t\geq 0\}$ (the *semigroup* axiom). We will usually write $\varphi^t(\cdot)$ instead of $\varphi(t,\cdot)$. In particular, we study semiflows generated by continuous *vector fields* over M^n, that is, when φ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\varphi^\tau(p)|_{\tau=t}=X(\varphi^t(p)),\quad \forall(t,p)\in\mathbb{R}_{\geq 0}\times M^n,\tag{1.1}$$

where X is a continuous section, that is, the map $X:M^n\to TM^n$ is continuous and satisfies $\pi\circ X=\mathrm{id}_{M^n}$ for π the canonical projection $(p,v)\mapsto\pi(p,v)=p$. When $M^n=\mathbb{R}^n$, then, the identification $T\mathbb{R}^n\simeq\mathbb{R}^n\times\mathbb{R}^n$ results in being able to discuss vector fields as simply self-maps of $\mathbb{R}^n$. On $\mathbb{R}^n$, we will work a lot with the "*canonical*" differential equation $\dot{x}=-x$, but also with the identity map $x\mapsto x$, thus, to avoid notational confusion, we denote the corresponding canonical vector field by $-\partial_x$, and not $-\mathrm{id}_{\mathbb{R}^n}$.

The focus on *semi*flows instead of flows allows us to look at sufficiently regular *discontinuous* vector fields as well. This is relevant to control theory, as the introduction of feedback usually results in a closed-loop vector field that cannot be assumed to be continuous (*e.g.*, think of optimal control[2]). This

[1] The reader that is unfamiliar with the notion of *orientation* is first directed to [GP10; Lee12] to see orientations in the smooth case. The topological definition relies on algebraic topology, see for instance [Hat02, Sec. 3.3].

[2] One might also think of topological obstructions, however, as will become clear below,

choice of setting is also motivated by other recent work. For instance, the main research question in [OJ24] is: "*Given an exponentially stable optimization algorithm, can it be modified to obtain a finite/fixed-time stable algorithm?*". They provide some sufficient conditions, we will show that a more generic, yet less constructive viewpoint is also possible.

Let X be some vector field on $\mathbb{R}^n$, possibly discontinuous. To study X, we usually pass to some *differential inclusion*

$$\dot{x} \in F(x), \tag{1.2}$$

where the set-valued map $F : \mathbb{R}^n \to 2^{\mathbb{R}^n}$, with 2^X denoting the *power set* of X, is in some precise sense related to X. The intuition is to pass from a irregular single-valued map, to a more regular set-valued map that contains the original behaviour.

Let λ^d denote the Lebesgue measure on $\mathbb{R}^d$, then, solutions to these differential inclusions, are *absolutely continuous* curves $\xi \in \mathrm{AC}(\mathcal{I} \subseteq \mathbb{R}; \mathbb{R}^n)$ such that $\dot{\xi}(t) \in F(\xi(t))$ for λ^1-*a.e.* $t \in \mathcal{I}$. Typically, F is assumed to be upper semi-continuous and compact, convex valued. With those assumptions in mind, then, under mild conditions on X, a valuable solution framework follows by applying *Filippov's operator* $\mathcal{F}$, that is,

$$x \mapsto \mathcal{F}[X](x) := \bigcap_{\delta>0} \bigcap_{N\in\{A\subset\mathbb{R}^n : \lambda^n(A)=0\}} \overline{\mathrm{conv}}\, X\left(\{x \in \mathbb{R}^n : \|x\|_2 < \delta\} \setminus N\right).$$

Then, solutions to

$$\dot{x} \in \mathcal{F}[X](x) \tag{1.3}$$

are understood to be "*generalized*" solutions to $\dot{x} = X(x)$, usually called *Filippov solutions*, *i.e.*, solutions to (1.3) are *absolutely continuous* curves $\xi : \mathcal{I} \to \mathbb{R}^n$ such that $\dot{\xi}(t) \in \mathcal{F}[X](\xi(t))$ for λ^1-a.e. $t \in \mathcal{I}$.

For more on differential inclusions and discontinuous dynamical systems, we point the reader to [Fil88], [BR05, Ch. 1] and [Cor08].

1.3 Stability To characterize stability under a differential inclusion (1.2), we need a few concepts. Starting with the regular case, for simplicity, let the vector field F be single-valued and smooth. In that case, F generates a flow, denoted $\varphi(\cdot; F)$. A point $x^\star \in \mathbb{R}^n$ is an *equilibrium point* of F when $F(x^\star) = 0$ or equivalently $\varphi^t(0; F) = 0\ \forall t \in \mathbb{R}$. We will set $x^\star$ to be 0, unless stated otherwise. Then, 0 is said to be *globally asymptotically stable* (GAS) (under F) when

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the type of discontinuities we consider do not allow for overcoming those obstructions, in general [Rya94].

($s.i$) 0 is *Lyapunov stable*, that is, for any open neighbourhood $U_\varepsilon \ni 0$ there is an open set $U_\delta \subseteq U_\varepsilon$ such that $\varphi^t(U_\delta; F) \subseteq U_\varepsilon$ $\forall t \in \mathbb{R}_{\geq 0}$;

($s.ii$) 0 is *globally attractive*, that is, $\lim_{t\to+\infty} \varphi^t(x_0; F) = 0$ for all $x_0 \in \mathbb{R}^n$.

Remark 1.2 (On global stability)**.** Although we work with $\mathbb{R}^n$, or $M^n \simeq_d \mathbb{R}^n$, we take a topological approach akin to [BH06] and hence the definition of GAS is as above. When working with metrics, one typically says more, *e.g.*, see [Wil69; LSW96], there, a metric is used to characterize how Lyapunov stability can be truly turned non-local (*e.g.*, let $B_r(x)$ be some r-metric ball at x, then, there is a some $\mathcal{K}_\infty$ function δ such that for any ε we have that $\varphi^t(B_{\delta(\varepsilon)}(x)) \subseteq \varphi^t(B_\varepsilon(x))$ $\forall t \geq 0$.) In general these definitions are not equivalent, however, for being GAS on $\mathbb{R}^n$, they are [ABB97]. ◦

Due to the work of Lyapunov [Lia92], we know that to reason about stability, it is worthwhile to look for "*potential functions*" that capture stability, illustrated by the fact that his theory effectively replaced the definitions of stability. Specially, we look for a function $V \in C^\infty(\mathbb{R}^n; \mathbb{R}_{\geq 0})$, satisfying the following properties:

($V.i$) $V(x) > 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$ and $V(0) = 0$;

($V.ii$) $\langle \nabla V(x), F(x) \rangle < 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$;

($V.iii$) V is *radially unbounded*, that is, $V(x) \to +\infty$ for $\|x\|_2 \to +\infty$.

Property ($V.iii$) implies sublevel set compactness and is sometimes referred to as *weak coercivity*. We call such a function a (smooth, strict and proper) Lyapunov function (with respect to the pair $(F, 0)$). This note is all about GAS, so we will omit "*strict*" and "*proper*" from now on. Then, based on converse theory by Massera, Kurzweil, Wilson and several others [Mas56; Kur63; Wil69; FP19], we can appeal to the celebrated theorem stating that 0 is GAS if and only if there is a (corresponding) smooth Lyapunov function [BR05, Thm. 2.4].

A generalization of the above to differential inclusions (1.2) is as follows.

Definition 1.1 (Strong Lyapunov pairs [CLS98, Def. 1.1])**.** *A pair of functions $(V, W) \in C^0(\mathbb{R}^n; \mathbb{R}_{\geq 0})$, with V being C^∞-smooth on $\mathbb{R}^n$ and W being C^∞-smooth on $\mathbb{R}^n \setminus \{0\}$, is said to be a C^∞-smooth strong Lyapunov pair for the vector field F as in (1.2), provided that F is upper semi-continuous and compact, convex valued, plus, the following conditions hold:*

(i) $V(x) > 0$ and $W(x) > 0$ on $\mathbb{R}^n \setminus \{0\}$, with $V(0) = 0$;

(ii) The sublevel sets of V are compact; and

(iii) $\max_{v\in F(x)}\langle \nabla V(x), v\rangle \leq -W(x)$ *on* $\mathbb{R}^n\setminus\{0\}$.

The preposition "*strong*" comes from the fact that we look at all solutions satisfying (1.2). Similarly, one could require that at least one solution is stable and construct a "*weak*" version of Definition 1.1.

For further references, and generalizations of Lyapunov's stability theory, we point the reader to [BS70; Son98; BR05; BH06; GST12].

Before closing this section, we explicitly illustrate why working with C^∞ or even C^0 vector fields is arguably overly restrictive when interested in qualitative stability questions. We start with a simple example.

Example 1.3 (A semiflow corresponding to a vector field with bounded discontinuities)**.** Consider the following *discontinuous* vector field on $\mathbb{R}$:

$$\dot{x} = X_1(x) := -\mathrm{sgn}(x) := \begin{cases} 1 & x<0 \\ 0 & x=0 \\ -1 & x>0 \end{cases}. \tag{1.4}$$

Now, consider the map $\varphi_1 : \mathbb{R}_{\geq 0}\times\mathbb{R}\to\mathbb{R}$ defined through

$$(t,x)\mapsto \varphi_1^t(x) := \begin{cases} \min\{0, x+t\} & x<0 \\ 0 & x=0 \\ \max\{0,x-t\} & x>0 \end{cases}. \tag{1.5}$$

One can check that φ is a global semiflow, describing a solution (*e.g.*, in the sense of Carathéodory, Krasovskii or Filippov [Fil88]) to (1.4)—a discontinuous vector field. In particular, note that $t\mapsto\varphi_1^t(x)\in \mathrm{AC}$ since $\varphi_1^t(x) = \varphi_1^0(x)+\int_0^t -\mathrm{sgn}(\varphi_1^\tau(x))\mathrm{d}\tau$ for any $t\geq 0$. Regarding stability, consider the C^∞ Lyapunov function $x\mapsto V_1(x) := \frac{1}{2}x^2$ and find that $\nabla V_1(x)X_1(x) = -|x| < 0$ on $\mathbb{R}\setminus\{0\}$ (consider a Huber loss to find a valid function W_1 in the sense of Definition 1.1). This already shows that the existence of a smooth Lyapunov function, asserting that the origin is GAS, does not imply the existence of a *flow*, nor does it imply that convergence to 0 is merely *asymptotic*. Now let $X_2(x) := -\partial_x$ and define $[0,1]\ni s\mapsto X(\cdot;s) := (1-s)X_1 + sX_2$. Then, for any $s\in[0,1]$ we have that $\nabla V(x)X(x;s) < 0$ on $\mathbb{R}\setminus\{0\}$. We know that the flow corresponding to X_2 is $(t,x)\mapsto\varphi_2^t(x) := e^{-t}x$. Better yet, by direct integration, we find that a global semiflow corresponding to $X(\cdot;s)$ becomes

$$(t,x)\mapsto \varphi^t(x;s) := \begin{cases} \min\{0, e^{-st}x + (1-e^{-st})(1-s)/s\} & x<0 \\ 0 & x=0 \\ \max\{0, e^{-st}x + (e^{-st}-1)(1-s)/s\} & x>0 \end{cases}.$$

Indeed, $s \mapsto \varphi(\cdot; s)$ parametrizes a homotopy, along global semiflows such that 0 is GAS, from φ_1 to φ_2. Regarding $\lim_{s\to 0^+}\varphi(\cdot; s)$, use that $e^{-st} = 1 - st + \sum_{n=2}^{\infty}((-st)^n/n!)$ to recover (1.5). It is interesting to note that although (1.5) is the unique (AC) solution to $\dot{x} \in \mathcal{F}[X_1](x)$, the solution to $\dot{x} \in \mathcal{F}[-X_1](x)$ is not unique, stability is key. ○

Next, we provide an example that will appear later.

Example 1.4 (An irregular gradient flow)**.** Let $\gamma \in \mathcal{K}_\infty$ be smooth on $(0, +\infty)$ and such that $\gamma(s)/\gamma'(s) \geq s$ (*e.g.*, $s \mapsto \gamma(s) = s^{(1/2)}$). Now consider the function $x \mapsto V_\gamma(x) := \gamma(\|x\|_2)$ on $\mathbb{R}^n$, which is C^∞ on $\mathbb{R}^n \setminus \{0\}$. Next, construct the vector field

$$\dot{x} = X_3(x) := \begin{cases} -\nabla V_\gamma(x) & x \neq 0 \\ 0 & x = 0 \end{cases}. \tag{1.6}$$

We cannot immediately appeal to Filippov's framework as X_3 is not necessarily a bounded operator. However, since $-\nabla V_\gamma(x) = -\gamma'(\|x\|_2)x/\|x\|_2$ we can study solutions to (1.6) directly. To that end, decompose $x \in \mathbb{R}^n \setminus \{0\}$ as $x = \|x\|_2 \cdot x/\|x\|_2 =: r \cdot u$. It readily follows that $\dot{r} = -\gamma'(r)$ while $\dot{u} = 0$. In general $\gamma'(s) > 0\ \forall s \geq 0$, need not be true, but suppose this is true for our choice of γ, *e.g.*, pick again $s \mapsto \gamma(s) = s^{1/2}$. Now, suppose that $1/\gamma'(s)$ is continuous and of class $\mathcal{K}$ on $\mathbb{R}_{\geq 0}$, then, we can define $\Gamma(r) := \int_0^r 1/\gamma'(\rho)\mathrm{d}\rho$, which is now of class $\mathcal{K}_\infty$ and hence invertible on $\mathbb{R}_{\geq 0}$. Under the aforementioned assumptions, we can define a semiflow corresponding to X_3 via (*e.g.*, derived via the inverse function theorem):

$$(t, x) \mapsto \varphi_3^t(x) := \begin{cases} \min\{0, \Gamma^{-1}(\Gamma(\|x\|_2) - t)\}x/\|x\|_2 & \|x\|_2 \geq \Gamma^{-1}(t) \\ 0 & \text{else} \end{cases}.$$

We emphasize that $\lim_{x\to 0}\varphi_3^t(x) = 0$ for any $t \geq 0$, which follows for $t = 0$ from $\varphi_3^0 = \mathrm{id}_{\mathbb{R}^n}$, whereas for $t > 0$ we have that $\min\{0, \Gamma^{-1}(\Gamma(\|x\|_2) - t)\} = 0$ for all $\|x\|_2 \leq \Gamma^{-1}(t)$. Note, the latter is inherent to the definition of Γ. ○

2 Further comments on related work Recently, we showed that when the origin $0 \in \mathbb{R}^n$ is GAS under a continuous vector field X, and if this can be asserted using a C^1 *convex* Lyapunov function V, then, X is straight-line homotopic to $-\partial_x$, such that the origin remains GAS along the homotopy [JS24]. This is a convenient result, but clearly not a general one.

Earlier, the following homotopy on the level of vector fields, reminiscent of *Alexanders's trick*, appeared in several works (*e.g.*, see [Rya94, p. 1603], [Son98, Thm. 21], [Cor07, p. 291] and [JM23, Ex. 3.4]):

$$(s,x) \mapsto H(s,x) := \begin{cases} X(x) & \text{if } s=0 \\ -x & \text{if } s=1 \\ \frac{1}{s}(\varphi^{s/(1-s)}(x;X)-x) & \text{if } s\in(0,1) \end{cases} . \tag{2.1}$$

Unfortunately, for (2.1), $0 \in \mathbb{R}^n$ is not known to be GAS along the path $[0,1] \ni s \mapsto H(s,x)$. The scalar and linear cases can be understood, however.

Example 2.1 (Stability-preserving homotopies for $n=1$)**.** As for any $t \geq 0$ the only fixed point of $\varphi^t(\cdot;X)$ is 0, it follows that for $n=1$, the homotopy (2.1) preserves stability (globally), simply because the sign of $x \mapsto H(s,x)$ cannot flip for otherwise, $\varphi^{s/(1-s)}(x;X) = x$ must hold for some $s \in (0,1)$ and $x \neq 0$. ○

Example 2.2 (Stability preserving homotopies for linear dynamical systems)**.** Let $0 \in \mathbb{R}^n$ be GAS under $\dot{x} = X(x) := Ax$, for $A =: TJT^{-1}$ the Jordan form decomposition of A. Then it readily follows that $\frac{1}{s}(\varphi^{s/(1-s)}(x;X)-x) = \frac{1}{s}T(e^{s/(1-s)J}-I_n)T^{-1}x$ such that stability is preserved throughout the homotopy (2.1). ○

Either way, the homotopy (2.1) does already show that there is *a* homotopy that does not introduce new equilibrium points, along the homotopy. Indeed, this has been generalized recently by Kvalheim to compact attractors on manifolds [Kva23]. However, all of this is not enough to conclude on the existence of a homotopy that preserves stability.

On the other hand, it is known that there is no reason why stability *must* be preserved along such a homotopy. In the spirit of [EM02, Sec. 4.1], consider the following family of linear vector fields on $\mathbb{R}^2$:

$$\dot{x} = X(x;s) := R(s)x, \quad [0,1] \ni s \mapsto R(s) := \begin{pmatrix} \cos(s\pi) & -\sin(s\pi) \\ \sin(s\pi) & \cos(s\pi) \end{pmatrix},$$

parametrizing a homotopy from $X(\cdot;0) = \partial_x$ to $X(\cdot;1) = -\partial_x$ along non-vanishing vector fields on $\mathbb{R}^n \setminus \{0\}$. Note that indeed, since $n=2$, the (Hopf) indices [Mil65, p. 32] of these two "*qualitatively opposite*" vector fields are equal, that is, $\mathrm{ind}_0(\partial_x) = 1^n = (-1)^n = \mathrm{ind}_0(-\partial_x)$. It is precisely this weakness of existing homotopy-invariants that we eventually hope to overcome by studying more restrictive equivalence classes.

3 Stability preserving homotopies

We first consider coordinates, that is, $0 \in \mathbb{R}^n$ being (strongly) GAS under some appropriately regular vector field on $\mathbb{R}^n$.

Assumption 3.1 (Vector field regularity on $\mathbb{R}^n$)**.** *The vector field X is a measurable map, possibly set-valued at 0, i.e., $0 \mapsto X(0) \in 2^{\mathbb{R}^n}$, with X being a bounded operator on $\mathbb{R}^n$ and locally Lipschitz away from 0.*

It is known that under the conditions of Assumption 3.1, $\mathcal{F}[X]$ is upper semi-continuous and compact, convex valued, which allows for a *smooth* converse Lyapunov theory, *e.g.*, see the discussions[3] in [CLS98].

Proposition 3.2 (Strong global asymptotic stability and homotopic semiflows on $\mathbb{R}^n$)**.** *Let $0 \in \mathbb{R}^n$, for $n \neq 5$, be strongly GAS, in the sense of Filippov, under $\dot{x} \in \mathcal{F}[X](x)$, with X satisfying Assumption 3.1. Then,*

(i) for any $t \geq 0$ we have that the time-t map of the semiflow generated by $\mathcal{F}[X]$, that is, $\varphi^t(\cdot; \mathcal{F}[X])$, is homotopic to $\varphi^t(\cdot; -\partial_x)$, along time-$t$ maps corresponding to semiflows such that 0 is strongly GAS; and

(ii) in particular, the semiflow $\varphi(\cdot; \mathcal{F}[X])$ is homotopic to $\varphi(\cdot; -\partial_x)$, along semiflows that preserve 0 being strongly GAS.

Proof. The proof proceeds in 5 steps. First we show that the time-t map $\varphi^t(\cdot; \mathcal{F}[X])$, corresponding to $\dot{x} \in \mathcal{F}[X](x)$, can be homotoped along semiflows to a gradient flow $\varphi^t(\cdot; -\nabla V)$, such that along the homotopy the origin remains strongly GAS. In Step (ii), by exploiting symmetry we show that a minor extension of [GSW99], allows for showing that for any $\gamma \in \mathcal{K}_\infty$ there is a $T \in \mathrm{Homeo}^+(\mathbb{R}^n; \mathbb{R}^n)$ (not just $T \in \mathrm{Homeo}(\mathbb{R}^n; \mathbb{R}^n)$) such that $V(T^{-1}(x)) = \gamma(\|x\|_2)$. In particular, it follows that T is homotopic to $\mathrm{id}_{\mathbb{R}^n}$ along a continuous path in $\mathrm{Homeo}^+(\mathbb{R}^n; \mathbb{R}^n)$. Then, in Step (iii), we use this map T to homotope $\varphi^t(\cdot; -\nabla V)$ to a semiflow, denoted $\widetilde{\varphi}$, that has $V \circ T^{-1}$ as its Lyapunov function. Again, 0 remains strongly GAS along the path. Next, we show in Step (iv) that $\widetilde{\varphi}^t$ can be homotoped to the time-t map $\varphi^t(\cdot; -\partial_x)$. At last, we combine the above and conclude in Step (v).

(i) Since 0 is strongly GAS under $\dot{x} \in \mathcal{F}[X](x)$ and X satisfies Assumption 3.1, there is a Lyapunov pair $V \in C^\infty(\mathbb{R}^n; \mathbb{R}_{\geq 0})$ and $W \in C^\infty(\mathbb{R}^n \setminus \{0\}; \mathbb{R}_{\geq 0})$, that certifies stability of $0 \in \mathbb{R}^n$ [CLS98, Thm. 1.3], under any Filippov solution to (1.2). In particular, we can construct the homotopy $H : [0,1] \times \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$ through non-vanishing vector fields, defined by $(s, x) \mapsto H(s, x) := (1-s)\mathcal{F}[X](x) - s\nabla V(x)$. Then, as for any

[3] To add, indeed, the existence of a smooth Lyapunov function for a discontinuous vector field relates to robustness [GST12] (*e.g.*, interpret Definition 1.1 as stability prevailing under all perturbations admissible through the inclusion) and of course to the topology of the space and attractor (*e.g.*, there cannot be a single smooth Lyapunov function on the circle $\mathbb{S}^1$ that asserts some point $p^\star \in \mathbb{S}^1$ is GAS under some (discontinuous) vector field X on $\mathbb{S}^1$).

$s \in [0,1]$ we have that $(1-s)\mathcal{F}[X] - s\nabla V$ also satisfies Assumption 3.1, plus

$$\langle \nabla V(x), H(s,x)\rangle \leq -(1-s)W(x) - s\langle \nabla V(x), \nabla V(x)\rangle,\ \forall x \in \mathbb{R}^n \setminus \{0\},$$

we find that 0 is strongly GAS, in the sense of Filippov, under $\dot{x} \in (1-s)\mathcal{F}[X](x) - s\nabla V(x)$ for any fixed $s \in [0,1]$. Now, to construct a homotopy on the level of semiflows, we follow Hartman [Har02, Ch. V, p. 93] and consider the following extended system Σ_H on $[0,1]\times\mathbb{R}^n$:

$$\Sigma_H : \begin{cases} \dot{s} = 0, \\ \dot{x} \in (1-s)\mathcal{F}[X](x) - s\nabla V(x). \end{cases}$$

Observe the following, $0 \in \mathbb{R}^n$ is strongly GAS under $\dot{x} \in (1-s)\mathcal{F}[X](x) - s\nabla V(x)$ for any fixed $s \in [0,1]$, which we can assert by means of the pair $(V, (1-s)W + s\|\nabla V\|_2^2)$. Therefore, as V has compact sublevel sets [CLS98, (L2)], any Filippov solution to Σ_H is defined for all $t \geq 0$, *e.g.*, see [Kha02, Thm. 3.3] for the standard ODE case. Now since $(s,x) \mapsto H(s,x)$ is locally Lipschitz away from 0, we can appeal to the *Picard-Lindelöf theorem* [Tes12, Thm. 2.2] and hence, by strong asymptotic stability [CLS98, Def. 2.1] *cf.* Example 1.3, any solution to Σ_H is also unique, which allows us to define the time-t map $(s,x) \mapsto \varphi^t((s,x);\Sigma_H)$ for any $t \geq 0$. In its turn, by considering the path $[0,1] \ni s \mapsto \varphi^t((s,\cdot);\Sigma_H)$, we see that this time-$t$ map φ^t defines a homotopy from the semiflow under $\mathcal{F}[X]$ to the (semi)flow under $-\nabla V$. Specifically, we have that $\varphi^t((0,x);\Sigma_H) = (0,\varphi^t(x;\mathcal{F}[X]))$, $\varphi^t((1,x);\Sigma_H) = (1,\varphi^t(x;-\nabla V))$ and $\varphi^t((s,0);\Sigma_H) = (s,0)$ $\forall s \in [0,1]$. Now, for any fixed $s \in [0,1]$, overload notation and define the time-t map $\varphi^t(\cdot;(s,\Sigma_H)) := \pi_{2:n+1} \circ \varphi^t((s,\cdot);\Sigma_H)$, for $\pi_{2:n+1}$ the projection on the last n coordinates. It follows that this map checks out as a semiflow, since $\pi_{2:n+1}$ is continuous and for any $x \in \mathbb{R}^n$ we have that

$$(i)\ \varphi^0(x;(s,\Sigma_H)) = x;\ \text{and}$$
$$(ii)\varphi^{t_2}(\varphi^{t_1}(x;(s,\Sigma_H));(s,\Sigma)) = \varphi^{t_2+t_1}(x;(s,\Sigma_H)), \quad \forall t_1,t_2 \geq 0.$$

(ii) Grüne, Wirth and Sontag showed that for our Lyapunov function V at hand, we can always find a $T \in \mathrm{Homeo}(\mathbb{R}^n;\mathbb{R}^n)$, with $T(0)=0$, such that $V(T^{-1}(x)) = \gamma(\|x\|_2)$, for some $\gamma \in \mathcal{K}_\infty$, smooth on $(0,+\infty)$ [GSW99, Prop. 1]. We recall their construction of this map. They define $(t,x) \mapsto \psi(t,x)$ to be the (local) flow with respect to the ODE

$$\dot{x} = \frac{\nabla V(x)}{\|\nabla V(x)\|_2^2}.$$

It follows that, on the domain of ψ, $V(\psi(t,x)) = V(x) + t$. Now fix some $c > 0$, then they define the map $\pi_c : \mathbb{R}^n \setminus \{0\} \to V^{-1}(c)$ by $x \mapsto \pi_c(x) :=$

$\psi(c - V(x), x)$, that is, starting from $x \in \mathbb{R}^n \setminus \{0\}$ you flow—either backward or forward—along ψ until you hit $V^{-1}(c)$. Then, due to initial work by Wilson [WJ67], we know that $V^{-1}(c) \simeq_h \mathbb{S}^{n-1}$, however, now—that is, after [GSW99] was published, the resolution of the Poincaré conjecture[4] implies there must be a homeomorphism $S : V^{-1}(c) \to \mathbb{S}^{n-1}$, for any $n \geq 1$. Next, define the map $Q := S \circ \pi_c : \mathbb{R}^n \setminus \{0\} \to \mathbb{S}^{n-1}$ and eventually the map $T : \mathbb{R}^n \to \mathbb{R}^n$ by

$$x \mapsto T(x) := \begin{cases} \gamma^{-1}(V(x))Q(x) & \forall\, x \in \mathbb{R}^n \setminus \{0\} \\ 0 & \text{otherwise,} \end{cases}$$

Now, it turns out that the particular choice of the homeomorphism $S : V^{-1}(c) \to \mathbb{S}^{n-1}$ is irrelevant for the construction of T as in [GSW99, Prop. 1]. Hence, we simply do the following. If $T \in \mathrm{Homeo}^+(\mathbb{R}^n; \mathbb{R}^n)$, that is, if T preserves *orientation*, we are done. If not, we can always adjust T, for instance, we can compose S with a map reflecting a single coordinate, denoted by $\rho : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$, *e.g.*, $(x_1, x_2, \dots, x_n) \mapsto (x_1, x_2, \dots, -x_n)$, to enforce this (recall that $\deg(\rho) = -1$ [GP10, Ch. 3]), that is, we use $\rho \circ S$ instead of S. As T can always be chosen to be orientation-preserving, T (and equivalently T^{-1}) can assumed to be a *stable homeomorphism*[5] and hence T is (C^0) isotopic[6] to $\mathrm{id}_{\mathbb{R}^n}$, *e.g.*, see [Kir69] and [Moi13, Ch. 11].

(iii) First, we note that, if necessary, we can always pick the homotopy $\widetilde{H} : [0, 1] \times \mathbb{R}^n \to \mathbb{R}^n$ from $\mathrm{id}_{\mathbb{R}^n}$ to T^{-1} such that 0 is always mapped to 0 along the path in $\mathrm{Homeo}^+(\mathbb{R}^n; \mathbb{R}^n)$, *e.g.*, for a homeomorphism $K : \mathbb{R}^n \to \mathbb{R}^n$ such that $K(0) \neq 0$, consider the map $x \mapsto L(x) := K(x) - K(0)$, with $y \mapsto L^{-1}(y) = K^{-1}(y + K(0))$.

Now, recall that $0 \in \mathbb{R}^n$ is (strongly) GAS under $\varphi^t(\cdot; -\nabla V)$, that is, $V(\varphi^t(x; -\nabla V)) - V(\varphi^0(x; -\nabla V)) \leq -\int_0^t W_V(\varphi^\tau(x; -\nabla V))\mathrm{d}\tau < 0$ for all $t > 0$ and any $x \neq 0$. Here, W_V is such that (V, W_V) is a Lyapunov pair, note that in this specific case we can select $x \mapsto W_V(x) := \|\nabla V(x)\|_2^2$. Then, since $\widetilde{H}(s, \cdot)$ is a homeomorphism with $\widetilde{H}(s, 0) = 0$, we also have for any $t > 0$ and $x \neq 0$

[4]Specifically, Perelman provided the final step ($\mathbb{S}^3$) in proving the *generalized Poincaré conjecture* in Top. For some historical comments, see [Sti12].

[5]The *stable homeomorphism theorem*—stating that all orientation-preserving homeomorphisms of $\mathbb{R}^n$ are stable—connects to the *annulus theorem* (which was a longstanding conjecture) via the work of Brown and Gluck [BG64] and has a rich history, with the key steps in the proof being provided by Kirby [Kir69] and Quinn [Qui82]. We point the reader to the survey of Edwards in the edited book by Gordon and Kirby on 4-manifolds [Edw84].

[6]Indeed, as also mentioned in [GSW99], T cannot (always) be a diffeomorphism on $\mathbb{R}^n$, for otherwise we constrain ourselves to topologically conjugate systems.

that

$$\begin{aligned} V(\varphi^t(\widetilde{H}(s,x);-\nabla V)) - V(\varphi^0(\widetilde{H}(s,x);-\nabla V)) \leq \\ -\int_0^t W_V(\varphi^\tau(\widetilde{H}(s,x);-\nabla V))\mathrm{d}\tau < 0. \end{aligned} \tag{3.1}$$

Next, define the semiflow $\widetilde{\varphi}(\cdot;s)$ through the following topological conjugacy

$$\widetilde{H}(s,\cdot)\circ\widetilde{\varphi}^t(\cdot;s) = \varphi^t(\cdot;-\nabla V)\circ\widetilde{H}(s,\cdot).$$

Note that $\widetilde{\varphi}^t(\cdot;0) = \varphi^t(\cdot;-\nabla V)$ whereas $\widetilde{\varphi}^t(\cdot;1) = T\circ\varphi^t(\cdot;-\nabla V)\circ T^{-1}$. Also note that continuity of $[0,1]\ni s\mapsto\widetilde{\varphi}^t(\cdot;s)$, in particular, continuity of $[0,1]\ni s\mapsto\widetilde{H}(s,\cdot)^{-1}$, follows from $\mathrm{Homeo}^+(\mathbb{R}^n;\mathbb{R}^n)$ being a *topological group* [Are46] (endowed with the compact-open topology), *i.e.*, combine that the map $\mathrm{Homeo}^+(\mathbb{R}^n;\mathbb{R}^n)\ni h\mapsto h^{-1}$ is continuous with path-connectedness being preserved under a continuous map. Then, it follows from (3.1) that for any $t>0$ and $x\neq 0$ we have that

$$\begin{aligned} V(\widetilde{H}(s,\widetilde{\varphi}^t(x;s))) - V(\widetilde{H}(s,\widetilde{\varphi}^0(x;s))) \leq \\ -\int_0^t W_V(\widetilde{H}(s,\widetilde{\varphi}^\tau(x;s)))\mathrm{d}\tau < 0. \end{aligned} \tag{3.2}$$

Now, using (3.2), define the (parametric in $s\in[0,1]$) Lyapunov function $\widetilde{V}(\cdot;s)\in C^0(\mathbb{R}^n;\mathbb{R}_{\geq 0})$ through $x\mapsto\widetilde{V}(x;s):=V(\widetilde{H}(s,x))$ and similarly, define $\widetilde{W}_V$ through $x\mapsto\widetilde{W}_V(x;s):=W_V(\widetilde{H}(s,x))$ Hence, we have that for any $s\in[0,1]$ the following Lyapunov inequality holds: $\widetilde{V}(\widetilde{\varphi}^t(x;s);s)-\widetilde{V}(\widetilde{\varphi}^0(x;s);s)\leq -\int_0^t\widetilde{W}_V(\widetilde{\varphi}^\tau(x;s);s)\mathrm{d}\tau<0$, for all $t>0$ and any $x\neq 0$. Note that $x\mapsto\widetilde{V}(x;0)=V(x)$ whereas $x\mapsto\widetilde{V}(x;1)=V(T^{-1}(x))=\gamma(\|x\|_2)=:V_\gamma(x)$ and $x\mapsto\widetilde{W}_V(x;1)=W_V(T^{-1}(x))$. Also, as $\widetilde{H}(s,\cdot)\in\mathrm{Homeo}^+(\mathbb{R}^n;\mathbb{R}^n)$ fixes 0, any compact neighbourhood K of 0 is mapped to some compact neighbourhood $\widetilde{H}(s,K)$ of 0, hence, $\widetilde{V}(\cdot;s)$ will always have the required compact sublevel sets.

(iv) Although the path from $\mathrm{id}_{\mathbb{R}^n}$ to T^{-1} is merely through $\mathrm{Homeo}^+(\mathbb{R}^n;\mathbb{R}^n)$, it is known[7] that T can always be chosen to be diffeomorphic on $\mathbb{R}^n\setminus\{0\}$ for $n\neq 5$ [GSW99] (note that the composition with ρ does not change this).

[7] Some comments are in place. Perelman's resolution came after the paper by Grüne, Sontag and Wirth, the case $n=4$ ($\mathbb{S}^3$) is resolved by now. Despite some claims in the literature, to the best of our knowledge, the case $n=5$ is still open. We emphasize that the existence of these diffeomorphisms is largely due to Smale's *h-cobordism theorem*, as the generalized Poincaré conjecture is known to fail, in general, for Diff (e.g., due to the existence of *exotic spheres* [Mil56]).

Hence, the vector field corresponding to $\widetilde{\varphi}(\cdot;1)$, denoted $\widetilde{F}$, is readily well-defined on $\mathbb{R}^n\setminus\{0\}$, *i.e.*,

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\widetilde{\varphi}^{\tau}(x;1)\bigg|_{\tau=0} = -\mathrm{D}T(T^{-1}(x))\nabla V(T^{-1}(x)) =: \widetilde{F}(x),\ \forall\, x\in\mathbb{R}^n\setminus\{0\}. \quad (3.3)$$

In general, $\widetilde{F}$ need not be continuous at 0, simply because $\mathrm{D}T$ need not be continuous at 0. However, γ can always be chosen such that T is C^1 on $\mathbb{R}^n$ with $\mathrm{D}T(0)=0$ [GSW99, Prop. 1].

Unfortunately, V_γ is by no means smooth at 0 for any choice of γ. In fact, the appropriate $\gamma\in\mathcal{K}_\infty$ from [GSW99, Prop. 1] to guarantee T is sufficiently regular is of the following form. Let $\alpha\in\mathcal{K}$ be smooth, define $h(r):=\int_0^r a(\tau)\mathrm{d}\tau$ and set $\gamma:=h^{-1}$. It readily follows that $\gamma'(r)=1/\alpha(\gamma(r))$ is smooth on $(0,+\infty)$, yet, $\lim_{r\to 0^+}\gamma'(r)=+\infty$, *e.g.*, $r\mapsto\gamma(r)=r^{1/2}$. However, note that on $\mathbb{R}^n\setminus\{0\}$ we have $\langle\nabla V_\gamma(x),\widetilde{F}(x)\rangle\leq -\widetilde{W}_V(x)$, which is equivalent to

$$\left\langle \gamma'(\|x\|_2)\frac{x}{\|x\|_2},\widetilde{F}(x)\right\rangle \leq -\widetilde{W}_V(x),\ \forall\, x\in\mathbb{R}^n\setminus\{0\}. \quad (3.4)$$

Thus, multiplying (3.4) by the function $a\in C^0(\mathbb{R}^n;\mathbb{R}_{\geq 0})$, defined through $x\mapsto a(x):=\|x\|_2/\gamma'(\|x\|_2)$, yields that $V_q\in C^\infty(\mathbb{R}^n;\mathbb{R}_{\geq 0})$ and $W_q\in C^\infty(\mathbb{R}^n\setminus\{0\};\mathbb{R}_{>0})$, defined by $x\mapsto V_q(x):=\frac{1}{2}\|x\|_2^2$ and $x\mapsto W_q(x):=-a(x)\widetilde{W}_V(x)$, comprise a Lyapunov pair for $\widetilde{F}$, also recall Example 1.4. Equivalently, one can observe that the level sets of V_γ are standard spheres.

Then, as in Step (i), we construct a homotopy through non-vanishing vector fields on $\mathbb{R}^n\setminus\{0\}$, in this case from $\widetilde{F}$ to $-\nabla V_q=-\partial_x$, with V_q being *a* Lyapunov function asserting that 0 is (strongly) GAS under $\widetilde{F}$. Specifically, construct the map $(s,x)\mapsto H_q(s,x):=(1-s)\widetilde{F}(x)-s\nabla V_q(x)$ (in this case for all $x\in\mathbb{R}^n$). Then, as for any $s\in[0,1]$ we have that $(1-s)\widetilde{F}-s\nabla V_q\in C^0(\mathbb{R}^n;\mathbb{R}^n)$, plus

$$\langle\nabla V_q(x),H_q(s,x)\rangle\leq -(1-s)W_q(x)-2sV_q(x),\ \forall\, x\in\mathbb{R}^n\setminus\{0\},$$

we find that 0 is (strongly) GAS under $\dot{x}=(1-s)\widetilde{F}(x)-s\nabla V_q(x)$ for any fixed $s\in[0,1]$. To pass to the level of semiflows we can proceed as in Step (i), that is, we can show there is homotopy from $\varphi^t(\cdot;\widetilde{F})$ to $\varphi^t(\cdot;-\partial_x)$ along semiflows, such that 0 is GAS.

(v) Since we have constructed the desired (i.e., preserving stability) homotopy between any time-t map $\varphi^t(\cdot;\mathcal{F}[X])$ to $\varphi^t(\cdot;-\partial_x)$, we just observe that all these maps are continuous in t so that we can conclude on the existence of the homotopy from the semiflow $\varphi(\cdot;\mathcal{F}[X])$ to $\varphi(\cdot;-\partial_x)$. $\square$

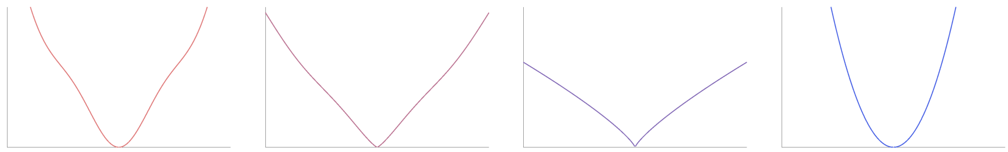

Figure 3.1: Example 3.2: on the left, the graph of V_i around $x = 0$; and on the right, the graph of V_q around $x = 0$. In between, two steps of the homotopy that connects V_i to V_q, through continuous function that preserve $x = 0$ being the global minimizer.

Proposition 3.2 is largely about the *existence* of a homotopy, less about the construction. Still, we provide some examples below.

Example 3.1 (Homotopic vector fields, preserving stability)**.** It is known that the following polynomial dynamical system does not admit a polynomial Lyapunov function asserting 0 is GAS [AKP11]. This 2-dimensional differential equation is defined by $\dot{x} = X_1(x) := (-x_1 + x_1x_2, -x_2)$. A non-polynomial C^∞ Lyapunov function that does assert 0 is GAS is given by $(x_1, x_2) \mapsto V_1(x_1, x_2) := (1/2)\log(1 + x_1^2) + (1/2)x_2^2$. To construct a homotopy from $\varphi(\cdot; X_1)$ to $\varphi(\cdot; -\partial_x)$, first construct the straight-line homotopy from X_1 to $-\nabla V_1$, i.e. $(1 - s)X_1 - s\nabla V_1$; stability is asserted throughout by V_1. Next, define $V(\cdot; s) := (1 - s)V_1 + sV_q$, for $s \in [0, 1]$ and $x \mapsto V_q(x) := (1/2)\|x\|_2^2$. It follows that, for any $s \in [0, 1]$, $\nabla V(x; s) = 0 \iff x = 0$, such that $[0, 1] \ni s \mapsto -\nabla V(\cdot; s)$ parametrizes a homotopy from $-\nabla V_1$ to $-\partial_x$, along vector fields that render 0 GAS, asserted by $V(\cdot; s)$. It is interesting to note that although X_1 does not admit a convex polynomial Lyapunov function, the negative gradient flow of the Lyapunov function corresponding to X_1 does *cf.* [JS24]. ◦

Example 3.2 (Homotopy from invexity to convexity)**.** Consider a coercive, invex function (*i.e.*, every critical point is a global minimizer) on $\mathbb{R}$ defined by $x \mapsto V_i(x) := (1/2)x^2 + (3/2)\sin(x)^2$. In the context of Proposition 3.2, we can find a map T such that $V(T^{-1}(x)) = \gamma(|x|) := \sqrt{|x|}$, and a path from T to $\mathrm{id}_{\mathbb{R}}$, as given by the homotopy

$$[0, 1] \times \mathbb{R} \ni (s, x) \mapsto H_i(s, x) := \operatorname{sgn}(x)\left(\frac{1 + s}{2}x^2 + (1 - s)\frac{3}{2}\sin(x)^2\right)^{2-(3/2)s}$$

with $H_i(0, \cdot) = T$ and $H_i(1, \cdot) = \mathrm{id}_{\mathbb{R}}$. In a second step one could consider the homotopy $(s, x) \mapsto x^{(1/2)+(3/2)s}$. Summarizing, we can construct the path from V_i to $x \mapsto V_q(x) := x^2$, along continuous functions such that 0 is the global minimizer throughout[8], see Figure 3.1. Evidently, for examples as simple as V_i one can find simpler homotopies. However, to the best of our knowledge, the mere existence of such a homotopy is still an open problem, in general.

However, Proposition 3.2 provides us with the existence of a homotopy from a smooth, proper Lyapunov function V on $\mathbb{R}^n$, asserting $0 \in \mathbb{R}^n$ is GAS, to $\mathbb{R}^n \ni x \mapsto \|x\|_2^2$, along *continuous*[9] Lyapunov functions that assert $0 \in \mathbb{R}^n$ is GAS along the homotopy. Differently put, we can find a homotopy from a coercive, invex function, to a convex function, such that along the homotopy we preserve the global minimizer. We believe this is of independent interest. ◦

As we allow for a class of discontinuous vector fields, Proposition 3.2 allows for a slight extension of the standard Hopf index, this has been pioneered by Gottlieb in earlier work, *e.g.*, see [GS95]. See also [CRT08] for a Conley index applicable to discontinuous vector fields and see [Kva21] for a hybrid version of the Poincaré-Hopf theorem.

We point out that one could omit Step (i) of the proof of Proposition 3.2, yet, as is also evident from Example 3.1, it is typically convenient to pass through a gradient flow. We exploit precisely this step in the proof of Theorem 3.4 below.

We also remark that the origin plays no particular role in Proposition 3.2, as it should. In fact, if X is such as in Proposition 3.2, yet, the equilibrium point is now arbitrary, we can still construct a homotopy between $\varphi(\cdot;\mathcal{F}[X])$ and $\varphi(\cdot;-\partial_x)$ such that along the homotopy *some* point is GAS. For instance, the following family of time-t maps parametrizes a homotopy between the flows corresponding to the ODEs $\dot{x} = -x$ and $\dot{x} = -(x-\bar{x})$:

$$(t,x) \mapsto \varphi^t(x;s) = e^{-t}x + s(I_n - e^{-t}I_n)\bar{x}, \quad s \in [0,1].$$

Indeed, $s\bar{x}$ is GAS along the homotopy (*e.g.*, consider the Lyapunov function $x \mapsto V(x;s) := \frac{1}{2}(x-s\bar{x})^2$).

Next, we generalize Proposition 3.2 to smooth manifolds, which is almost immediate as points being GAS heavily restrict the class of manifolds, *e.g.*, see [BS70, Ch. V.3]. To clarify terminology, an n-dimensional manifold M^n is said to be *ψ-diffeomorphic* to $\mathbb{R}^n$ when $\psi(M^n) = \mathbb{R}^n$ for a diffeomorphism ψ. We will always assume that our manifold is Hausdorff and second countable. Regarding our vector fields, we assume the following.

Assumption 3.3 (Vector field regularity on M^n)**.** *Given a C^∞ manifold M^n, the continuous vector field X on M^n is such that X gives rise to a continuous (global) flow $\varphi(\cdot;X) : \mathbb{R} \times M^n \to M^n$.*

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[8]For a simulation of this homotopy, see `wjongeneel.nl/figinvex.gif`.

[9]It is not evident, and currently unknown, whether smoothness can be preserved throughout the homotopy, see Step (iii) of the proof of Proposition 3.2.

Comparing to Proposition 3.2, we note that the vector fields under consideration in Theorem 3.4 (*i.e.*, Assumption 3.3) can be less regular indeed. To avoid too many technicalities, we refrain from those generalizations, however.

Theorem 3.4 (Stability preserving semiflows)**.** *Let M^n be a C^∞ manifold, for $n \neq 5$, and let $p^\star \in M^n$ be GAS under both the vector fields X and Y, with both X and Y satisfying Assumption 3.3. Then, the flow $\varphi(\cdot; X)$ is homotopic to $\varphi(\cdot; Y)$, along flows that preserve $p^\star$ being GAS.*

Proof. First, since $p^\star$ is GAS under X, there is $V \in C^\infty(M^n; \mathbb{R}_{\geq 0})$ such that $X(V) = L_X V < 0$ on $M^n \setminus \{p^\star\}$ [FP19]. Now we fix some Riemannian metric g, which always exists [Lee12, Ch. 13], such that we can define the Riemannian gradient $\operatorname{grad} V$ (omitting the dependence on g). It follows that $-\operatorname{grad} V(V) < 0$ on $M^n \setminus \{p^\star\}$ such that we can define the straight-line homotopy between X and $-\operatorname{grad} V$, preserving that $p^\star$ is GAS.

Due to smoothness and stability, we know that $-\operatorname{grad} V$ gives rise to a flow, denoted $\varphi(\cdot; -\operatorname{grad} V)$, on M^n. In fact, for any $t \in \mathbb{R}$, $p \mapsto \varphi^t(p; -\operatorname{grad} V)$ is a diffeomorphism. This implies in particular that M^n is ψ-diffeomorphic to $\mathbb{R}^n$ [WJ67, Thm. 2.2], for some diffeomorphism $\psi : M^n \to \mathbb{R}^n$.

Next, define the diffeomorphism $\widetilde{\psi}$ by $M^n \ni p \mapsto \widetilde{\psi}(p) := \psi(p) - \psi(p^\star) \in \mathbb{R}^n$, such that $\widetilde{\psi}(p^\star) = 0$, and consider $x := \widetilde{\psi}(p)$ such that in these new coordinates we have

$$\dot{x} = -\mathrm{D}\widetilde{\psi}(\widetilde{\psi}^{-1}(x))\operatorname{grad} V(\widetilde{\psi}^{-1}(x)) =: X(x)$$

Indeed, the vector field X meets precisely the criteria of Proposition 3.2. Then, observe that $\varphi^t(\cdot; -\operatorname{grad} V) = \widetilde{\psi}^{-1} \circ \varphi^t(\cdot; X) \circ \widetilde{\psi}$, which implies that $\varphi^t(\cdot; -\operatorname{grad} V)$ can be homotoped to $\widetilde{\psi}^{-1} \circ \varphi^t(\cdot; -\partial_x) \circ \widetilde{\psi}$ such that $p^\star$ remains GAS along the homotopy.

We can do the exact same for the vector field Y. Then, since $\widetilde{\psi}$ is independent of the vector fields X and Y we can conclude by the transitivity property of homotopies, that is, both the time-t maps under X and Y can be homotoped to $\widetilde{\psi}^{-1} \circ \varphi^t(\cdot; -\partial_x) \circ \widetilde{\psi}$.

□

Example 3.3 (Equilibria on the sphere)**.** Regarding examples of Theorem 3.4 one might think of rendering the South Pole S an attractor on $\mathbb{S}^2 \setminus \{N\}$. To that end, consider the vector fields X and Y on $\mathbb{R}^2$, defined through

$$\begin{aligned} X(x) &:= (-0.1x_1 - x_2)\partial_{x_1} + (x_1 - 0.1x_2)\partial_{x_2}, \\ Y(x) &:= (-x_1 - x_1 x_2^2)\partial_{x_1} + (-x_2 + x_1^2 x_2)\partial_{x_2}. \end{aligned}$$

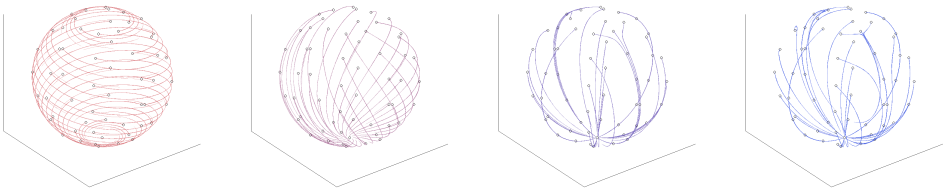

Figure 3.2: Example 3.3: on the left, some flow lines under $(\Pi_N^{-1})_* X$; and on the right, some flow lines under $(\Pi_N^{-1})_* Y$. In between, two steps of the homotopy that connects $(\Pi_N^{-1})_* X$ to $(\Pi_N^{-1})_* Y$, through vector fields that render S GAS on $\mathbb{S}^2 \setminus \{N\}$.

The origin is GAS under both X and Y (*e.g.*, consider the canonical quadratic Lyapunov function). Let Π_N be the stereographic projection from $\mathbb{S}^2 \setminus \{N\}$ to $\mathbb{R}^2$. Then, to transform X and Y to vector fields on $\mathbb{S}^2 \setminus \{N\}$, we construct the *pushforwards* $(\Pi_N^{-1})_* X$ and $(\Pi_N^{-1})_* Y$, see Figure 3.2. Exploiting this structure and our previous work [JS24], we can construct an explicit homotopy between these two vector fields that preserves stability of S on $\mathbb{S}^2 \setminus \{N\}$[10]. ◦

Remark 3.4 (On weaker notions of stability)**.** We note that, in general, one cannot relax global asymptotic stability to mere stability. A reason being that the (Hopf) index of corresponding equilibria under those vector fields is by no means fixed to $(-1)^n$ [KZ84, Sec. 52] (for n the dimension of the state space), that is, since this index is homotopy invariant, if the indices are different, no homotopy exists between them. This also means that these equilibria cannot just be homotoped to equilibria that are GAS. Interestingly, Krasnosel'skiĭ and Zabreĭko considered precisely that question: "*We now turn to the following question: Consider an autonomous system such that zero is an equilibrium state which is only known to be Ljapunov-stable. Is it always possible to deform this system into a system such that zero is an asymptotically stable equilibrium point?*" [KZ84, p. 342]. ◦

Next, we briefly comment on the particular class of semiflows we consider.

Remark 3.5 (On flows generated by vector fields)**.** Converse Lyapunov theory for flows is intimately connected to vector fields, *e.g.*, see [FP19]. For instance, techniques to smooth a continuous Lyapunov function for a flow naturally result in a smooth function that allows for asserting stability under the vector field that generates this flow. Secondly, there are flows—not generated by a continuous vector field—that fail to admit a smooth Lyapunov function [FP19, Sec. 7]. This motivated us to work with (semi)flows generated by vector fields. It is unclear if a generalization exists. Note, we are not talking about maps. ◦

[10] For a numerical simulation of the homotopy, see `wjongeneel.nl/figStereoS2.gif`. Note that the homotopy is through the canonical vector field indeed.

4 A view from optimal transport Let $0 \in \mathbb{R}^n$ be GAS under the smooth ODE $\dot{x} = X(x)$. As such, we know that there must be a smooth Lyapunov function V, asserting stability of 0. We can simply construct a straight-line homotopy from X to $-\nabla V$, *i.e.*, $[0,1] \ni s \mapsto (1-s)X - s\nabla V$ and preserve stability throughout (since V remains a valid Lyapunov function). Now it is tempting to believe that we should be able to homotope $x \mapsto V(x)$ into $x \mapsto V_q(x) := \frac{1}{2}\|x\|_2^2$ along "Lyapunov functions", specifically, along invex functions such that their negative gradient flow is well-defined and renders 0 GAS. By passing to semiflows instead of vector fields, Theorem 3.2 confirms this belief.

However, to refine our understanding towards a similar result for vector fields, we note that our desire relates to *optimal transport* (OT) in the following way. Let $\mu_0, \mu_1 \in \mathscr{P}(\mathbb{R}^n)$ be Borel probability measures on $\mathbb{R}^n$. Now suppose that $\mathrm{d}\mu_0 = \kappa_0 e^{-V}\mathrm{d}\lambda^n$ and $\mathrm{d}\mu_1 = \kappa_1 e^{-V_q}\mathrm{d}\lambda^n$, for κ_0, κ_1 normalization constants, λ^n the Lebesgue measure on $\mathbb{R}^n$ and e^{-V}, e^{-V_q} (unnormalized) densities corresponding to our Lyapunov functions. Can we transport μ_0 to μ_1 along sufficiently regular measures, preserving unimodality?

Suppose the answer is yes, then we can construct a path of densities $[0,1] \ni s \mapsto f(x;s)$, with $f(x;0) = e^{-V(x)}$ and $f(x;1) = e^{-V_q(x)}$. Now, if f is sufficiently regular, we can conclude that $\nabla \log f(x;s)$ is a smooth vector field, rendering 0 GAS, for each fixed s. Thus, understanding the transportation of measures can provide further insights. It is particularly interesting that the standard Gaussian measure results in the canonical ODE $\dot{x} = -x$.

In what follows we touch upon this viewpoint and show how OT and our homotopy questions are indeed intimately related.

The "*Monge formulation*", with a quadratic cost on $\mathbb{R}^n$, of OT is as follows, let $\mu, \nu \in \mathscr{P}_2(\mathbb{R}^n) := \{\mu \in \mathscr{P}(\mathbb{R}^n) : \int \|x\|_2^2 \mathrm{d}\mu(x) < +\infty\}$, then we would like to solve

$$\inf_{\{T : T\#\mu = \nu\}} \int_{\mathbb{R}^n} \|x - T(x)\|_2^2 \mathrm{d}\mu(x), \tag{4.1}$$

where $T\#\mu$ denotes the *pushforward* of μ, *i.e.*, $T\#\mu(A) = \mu(T^{-1}(A))$ for all Borel sets $A \subseteq \mathbb{R}^n$. Now, Brenier's seminal work [Bre91], showed that if $\mu \ll \lambda^n$, then, there is always a convex map $\phi : \mathbb{R}^n \to \mathbb{R}$, being μ-a.e. differentiable, such that $\nabla\phi\#\mu = \nu$ solves the Monge problem (4.1). In general, ϕ is not smooth and regularity of the optimal transport map is typically studied through the *Monge-Ampère equation*, *e.g.*, see [Phi13].

To illustrate what can already be said, we do a simple example.

Example 4.1 (Gaussian optimal transport)**.** Suppose we have two zero-mean Gaussian measures μ_0 and μ_1 on $\mathbb{R}^n$, with covariance matrices Σ_0 and Σ_1, respectively. In this case, the optimal transportation map, in the sense of (4.1), is simply $x \mapsto T(x) := Ax$, for $A := \Sigma_0^{-1/2}(\Sigma_0^{1/2}\Sigma_1\Sigma_0^{1/2})^{1/2}\Sigma_0^{-1/2}$, that is, $T\#\mu_0 = \mu_1$, *e.g.*, see [PC19, Rem. 2.31]. In fact, we can construct the interpolation $T(\cdot;s) := (1-s)\mathrm{id}_{\mathbb{R}^n} + sT$, with $s \in [0,1]$ and show that for $\mu(\cdot;s) := T(\cdot;s)\#\mu_0$ we have that

$$\Sigma(s) = \Sigma_0^{-1/2}\left((1-s)\Sigma_0 + s(\Sigma_0^{1/2}\Sigma_1\Sigma_0^{1/2})^{1/2}\right)^2 \Sigma_0^{-1/2} \succ 0.$$

Indeed, Brenier's map is simply $x \mapsto \phi(x;s) = (1-s)\frac{1}{2}\langle x, x\rangle + s\langle x, Ax\rangle$. Now consider the density $\rho(\cdot;s)$ defined through

$$\mathbb{R}^n \ni x \mapsto \rho(x;s) = \frac{1}{\sqrt{(2\pi)^n \det(\Sigma(s))}} e^{-\frac{1}{2}\langle x, \Sigma(s)^{-1}x\rangle}$$

First, note that $s \mapsto \Sigma(s)^{-1}$ is smooth (as captured by the Lie group structure of real invertible matrices, *e.g.*, see [DK99]). Then, set $V(\cdot;s) := -\log\rho(\cdot;s)$. Now suppose that $\Sigma_1 = I_n$. It follows that the family of ODEs $\dot{x} = -\nabla V(x;s)$, as parametrized by $s \in [0,1]$, comprises a homotopy from $\dot{x} = -\nabla V(x;0)$ to $\dot{x} = -x$, through vector fields such that 0 is GAS (*e.g.*, consider the Lyapunov function $x \mapsto \widetilde{V}(x;s) := \frac{1}{2}\langle x, \Sigma(s)^{-1}x\rangle$). ◦

Indeed, the result from Example (4.1) is already known in that sense that for vector fields equipped with convex Lyapunov functions—in particular, linear vector fields—we can construct a stability-preserving homotopy on the level of vector fields [JS24]. Note that the fact that we could exploit convexity fits in precisely with work by Caffarelli, *e.g.*, see [Caf00, Thm. 11].

5 Conclusion and future work

We have provided a first step towards addressing Conley's converse question in generality far beyond our previous work [JS24], yet, many open questions remain. Although directly working with flows has its benefits, *e.g.*, see [ADJ23], the main open problem is evidently the extension to vector fields and generic attractors. Several other open questions are as follows.

Open question 1: *prove or disprove that* (2.1) *preserves stability throughout the homotopy.*

Open question 2: *prove or disprove that Proposition 3.2 holds for $n = 5$. Note that a counterexample would disprove the generalized Poincaré conjecture in* Diff *for 4-dimensional spheres.*

Although we work with semiflows, we rely on vector fields generating them (see Remark 3.5). Removing this condition is non-trivial, illustrated by [FP19, Sec. 7], but also not futile, as illustrated by the simple example below.

Example 5.1 (An attractor on the mapping torus)**.** To construct an example of a semiflow that does not correspond to a vector field, one can appeal to maps with a "negative orientation", *e.g.*, see that a smooth vector field X always results in a flow $\varphi(\cdot;X)$ such that for any $t>0$ the diffeomorphism $\varphi^t(\cdot;X)$ is isotopic to the identity, and the identity map has positive orientation[11]. To that end, consider the map $\mathbb{R}\ni x\mapsto f(x):=-\alpha x$, for some $\alpha\in(0,1)$, and the define the mapping torus of $(\mathbb{R},f)$ through $M_f:=\{(x,t)\in\mathbb{R}\times\mathbb{R}_{\geq 0}:0\leq t\leq 1\}/\sim$ for $(x,1)\sim(f(x),0)$. Now, the *suspension* of f (under the ceiling function $c(x)\equiv 1$) is the semiflow $\varphi(\cdot;f)$ defined through $\varphi^t((x,s);f):=(f^n(x),s')$ where the pair (n,s') satisfies $n+s'=t+s$ with $0\leq s'\leq 1$, *e.g.*, see [BS02, Sec. 1.11]. For our choice of f it follows that the circle $\{0\}\times[0,1]/\sim$ is an attractor under $\varphi(\cdot;f)$ on M_f. ◦

Open question 3: *prove or disprove that Proposition 3.2 holds for all semiflows that render* 0 *GAS.*

We also consider relaxing Assumption 3.1, which is instrumental at the moment to be able to appeal to a converse, smooth Lyapunov theory. However, Example 1.4 already showed that the vector field need not be bounded at 0 for a semiflow to exist.

Open question 4: *relax Assumption 3.1 as far as possible.*

Section 4 just touched upon connections with optimal transport. Motivated by similar work in the context of geometry processing [SV19] (vector field interpolation), we believe this direction deserves a further study.

Open question 5: *elucidate what optimal transport can tell us about the existence of stability preserving homotopies on the level of vector fields, plus, what can we tell optimal transport?*

At last we point out that classical results typically tell us that certain maps, equivalence classes and so forth, exist. These results, however, rarely provide these objects explicitly. Recent work aims at finding algorithmic schemes to obtain these objects, may it be approximately, *e.g.*, see [BBK21]. Our work might also benefit from more explicit results, especially when applied in the context of optimization.

A Continuation and the Conley index

The existence of a homotopy on M^n is not particularly insightful by itself. More interesting is to consider classification up to homotopy, *e.g.*, Hopf's celebrated degree theorem states (in its simplest form) that all continuous maps from $\mathbb{S}^m$ to itself, are completely classified up to homotopy, via their (topological) degree, *e.g.*, see [Mil65, p. 51].

[11] Consider the homotopy $[0,1]\ni s\mapsto\varphi^{\tau(1-s)}$ from φ^τ to $\varphi^0=\mathrm{id}$.

The question that we try to address is motivated by better understanding to what extent Hopf's theorem extends to qualitatively equivalent classes of dynamical systems.

The most fruitful framework in this regard is Conley's theory [Con78]. It is outside the scope of this note to cover these ideas, but we briefly highlight it to elucidate the central question of this work.

In the context of semiflows, Conley's index theory can be set up as follows [Ryb87, Ch. 1]. Given a semiflow φ on M^n, then, $S \subseteq M^n$ is said to be an *isolated (positively) invariant set* when there is a compact set $K \subseteq M^n$, called an *isolating neighbourhood*, such that

$$S = \mathrm{I}(K, \varphi) := \{p \in K : \varphi^t(p) \in K \, \forall t \geq 0\} \subseteq \mathrm{int}(K).$$

Now, a pair of compact sets $(N, L) \subset M^n \times M^n$ is said to be an *index pair* for S when

($I.i$) $S = \mathrm{I}(\mathrm{cl}(N \setminus L), \varphi)$ and $N \setminus L$ is a neighbourhood of S;

($I.ii$) L is positively invariant in N; and

($I.iii$) L is an exit set for N.

Then, the (homotopy) *Conley index* of S is the homotopy type of the pointed (quotient) space $(N/L, [L])$. Importantly, the Conley index is independent of the choice of index pair. Towards a computational theory, one usually works with the homological definition instead.

Now, if some N can be chosen to be an isolating neighbourhood *throughout* a homotopy $[0, 1] \ni s \mapsto \varphi(\cdot; s)$, then, the Conley index is preserved along that homotopy and we speak of a *continuation* (of the Conley index). The existence of the homotopy implies continuation, but to what extent does an equivalent Conley index imply the existence of an index preserving homotopy? This is precisely the starting point of this note and this is why we speak of a "*converse question*".

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