# On Topological Equivalence in Linear Quadratic Optimal Control

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***Abstract*— Dynamical systems are topologically equivalent when their orbits can be mapped onto each other via a homeomorphic change of coordinates. We will show that in general, closed-loop systems resulting from Linear Quadratic Optimal Control problems are all topologically equivalent. As such, we provide new insights in structural "tuning" of controlled behaviour.**

## I. Introduction

Ever since its inception and celebration, the theory of optimal control also received critique. The cost is commonly scalar-valued, making the optimal control selection solely dependent on a single performance criteria, which limits practicality [1]. To better understand the theory, as being one of its pioneers, R. E. Kalman set out to understand the *inverse* problem [2], that is, given a control policy, does there exist an optimal control problem giving rise to this policy? To quote his motivation "*...discover general properties shared by all optimal control laws. We might be able to separate control laws which are optimal in some sense from those which are not optimal in any sense.*". In this work we add to this investigation, with an emphasis on the *controlled* behaviour.

In particular, we will consider the discrete-time Linear Quadratic Regulation (LQR) problem. This is a classical setting which made its appearance in many real-world systems. There, one usually encounters the notion of "*tuning*" the cost function such that the system is "*sufficiently*" stable. Success-stories of this tuning can be found throughout, with the catch being that there, linear feedback is designed for a *locally*-linear system, where tuning might be needed indeed. This note shows that, for the better or worse, if one *does* have a linear system how to change the closed-loop system behaviour *structurally*.

### A. Related work

To classify the behaviour of a controlled dynamical system we take a topological approach. The advantage is that we can greatly simplify the study and pass from a continuum of systems to just a set of classes which is usually finite. The topological classification of linear flows and maps was pioneered by Kuiper and co-workers [3]. A range of these ideas were later extended into the system theoretic direction [4]. There, the author expected that these concepts would "*become standard vocabulary among practitioners.*". Although this did not became a reality, some attention has been given to topological feedback linearization, *e.g.,* see [5], plus there is recent interest in structural stability in the context of systems biology [6]. We believe this lack of interest is in part due to the fact that topological classification seems to be rather coarse, plus, applications are usually not clear. In this note we hope to provide some concrete motivation and perhaps contribute to re-instigating this beautiful field.

As in the original work by Kuiper and Robbin, we focus on linear dynamical systems, but this time, in line with Kalman, driven by some optimal Linear Quadratic (LQ) regulator, or any other policy originating from the family of LQ optimal control problems. LQ theory is well-understood, especially in the context of classical engineering [7] and currently in the context of statistical reinforcement learning [8]–[10] and optimization [11]. In the context of adaptive control, several interesting topological results, with respect to the underlying model, are made in [12]–[14]. Topological insights in the *resulting* systems are less known, or at least, not described as much in the modern literature. We try to fill in this gap and provide a new interpretation of how one can change the dynamical behaviour, structurally, via selecting appropriate cost-matrices. To avoid confusion, we would like to stress that the vast majority of work on *topology* in the context of control, relates to *network* topology, which is *not* what this note is about.

### B. Contribution and outline

The main contribution of this note is to show that a well-known class of optimization problems have structurally equivalent minimizers, *i.e.*, in optimization parlance, without defining the class $\mathcal{F}$ and *equivalence relation* $\sim$, we have

$$\arg\min_{x\in\mathcal{X}} f_1(x) \sim \arg\min_{x\in\mathcal{X}} f_2(x) \quad \forall f_1, f_2 \in \mathcal{F}.$$

Specifically, we highlight that the most common class of Linear Quadratic (LQ) Optimal Control (OC) problems result in topologically equivalent closed-loop behaviour[1]. In particular, we show that by means of tuning the cost matrices, a bifurcation in the controlled system can only be induced by the introduction of cross-terms. Concurrently, building on [12], we see that a lot of structure of the underlying system is LQ feedback invariant. These observations have some implications, for example to reduce the dimension of the optimal control problem, to give a new interpretation of cross-terms in the cost or to preserve structure from a corresponding continuous-time problem. Although the arguments are simple, to the best of our knowledge, this is the first time they appear.

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[1]Preliminary arguments appeared in [15].

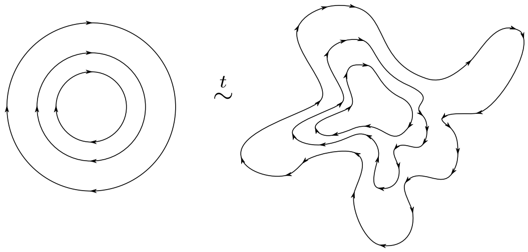


Fig. 1: We speak of *topological equivalence*, denoted $\overset{t}{\sim}$, when two phase portraits are homeomorphic.

*Notation:* We denote the real $n$-dimensional *General Linear group* by $\mathsf{GL}(n,\mathbb{R}) := \{A \in \mathbb{R}^{n\times n} : \det(A) \neq 0\}$. The group $\mathsf{GL}(n,\mathbb{R})$ can be written as $\mathsf{GL}^-(n,\mathbb{R}) \cup \mathsf{GL}^+(n,\mathbb{R})$, which is the disjoint union of two path-connected sets. Here, the superscript denotes the sign of the determinant, *e.g.,* $T \in \mathsf{GL}^+(n,\mathbb{R}) \iff \det(T) > 0$. A matrix $A \in \mathbb{C}^{n\times n}$ is *asymptotically stable* when $\rho(A) := \max_i |\lambda_i(A)| < 1$.

## II. Preliminaries

In this section we will introduce the notion of *topological equivalence* in dynamical systems and the concept of *linear optimal control* with *quadratic* cost functions.

### A. Topological equivalence in linear dynamical systems

This part mainly highlights the work of Kuiper and Robbin [3], [16], looking at dynamical systems of the form

$$x \mapsto f(x), \quad x \in \mathcal{V} \tag{1}$$

where the *time-one map* $f : \mathcal{V} \to \mathcal{V}$ is a linear endomorphism[2] over a finite-dimensional topological vector space $\mathcal{V}$. Then, we say that two dynamical systems are *topologically equivalent* when their phase-portraits are homeomorphic[3] [17],[18, Chapter 2] (see Figure 1). The purpose of this tool is to characterize classes of dynamical systems giving rise to *qualitatively* similar trajectories. This notion appears in the celebrated Hartman-Grobman theorem [17, Theorem 5.3, page 153] and is the key concept in Bifurcation theory [18], which studies precisely this qualitative change in dynamical systems. In fact, we speak of a *bifurcation* when a system, after some parameter change, is not (locally) topologically equivalent anymore to its initial configuration. The notion of topological equivalence has an explicit characterization in the discrete-time setting, there it coincides with the two time-one maps being *conjugates*.

**Definition II.1** (Topological equivalence)**.** *Two endomorphisms $f : \mathcal{V} \to \mathcal{V}$ and $g : \mathcal{W} \to \mathcal{W}$ over topological vector spaces $\mathcal{V}$ and $\mathcal{W}$ are* topologically equivalent (conjugate)*, denoted $f \overset{t}{\sim} g$, if and only if there exists a homeomorphism $\varphi : \mathcal{V} \to \mathcal{W}$ such that $g \circ \varphi = \varphi \circ f$, that is, the diagram*

$$\begin{tikzcd} \mathcal{V} \arrow[r, "f"] \arrow[d, "\varphi"'] & \mathcal{V} \arrow[d, "\varphi"] \\ \mathcal{W} \arrow[r, "g"] & \mathcal{W} \end{tikzcd} \tag{2}$$

*commutes.*

Instead of Definition II.1 one encounters the stronger notion of *linear equivalence* more often. Indeed, for any $T \in \mathsf{GL}(n,\mathbb{R})$ and $A \in \mathbb{R}^{n\times n}$ the diagram (2) commutes for $f(x) = Ax$, $g(y) = TAT^{-1}y$, *i.e.,* $\varphi(x) = Tx$. However, the quotient space under linear equivalence is still a continuum, whereas from a topological point of view, there are for example just 7 scalar systems [3, Proposition 1.5]. Hence, one can think of Definition II.1 as a weaker change of coordinates. However, one should merely assume that the map $\varphi$ is a homeomorphism, assuming $\varphi$ to be a diffeomorphism implies that $\varphi$ is linear [17, Proposition 6.1, page 43]. To clarify Definition II.1, examine the example from [3] given by $f(x) = 2x$ and $g(y) = 8y$. Although their eigenvalues are clearly different, *qualitatively*, $f$ and $g$ are the same. Indeed, $f \overset{t}{\sim} g$ since $\varphi(x) = x^3$ is the corresponding homeomorphism. Observe that although $\varphi \in C^\omega(\mathbb{R})$, the inverse $\varphi^{-1}(x) = \sqrt[3]{x} \in C^0(\mathbb{R})$. Then, Kuiper and Robbin [3] propose several conditions on the (generalized) eigenspaces of the linear endomorphims $f$ and $g$ to show topological equivalence. We will mainly focus on two of them; stability, but most and for all: *orientation*.

**Definition II.2** (Orientation of linear maps)**.** *We call a linear automorphism $f$ orientation preserving when the sign of the signed volume of the unit cube is invariant under the map $f$. This preservation (of orientation) is denoted by* $\mathrm{or}(f) = 1$*, otherwise* $\mathrm{or}(f) = -1$.

For example, given $f(x) := Fx$ and $g(y) := Gy$ with $F \in \mathsf{GL}^+(n,\mathbb{R})$ and $G \in \mathsf{GL}^-(n,\mathbb{R})$, then, $\mathrm{or}(f) = 1$ while $\mathrm{or}(g) = -1$. The intuitive reason why stability and orientation show up is as follows. Given time-one maps $f$ and $g$, the state-trajectories they induce are homeomorphic when there is a homeomorphism $\varphi$ such that $f = \varphi \circ g \circ \varphi^{-1}$. The link with stability follows from the observation that this definition implies that $f^n = \varphi \circ g^n \circ \varphi^{-1}$ must hold for any $n$, that is, the direction of time is enforced. As $\varphi$ will be either orientation preserving or reversing, Definition II.1 implies that $\mathrm{or}(f) = \mathrm{or}(g)$. In fact, orientation is a *topological invariant* [19, Chapter 6, 10], such that for two automorphisms $f$ and $g$, $f \overset{t}{\sim} g$, only if $\mathrm{or}(f) = \mathrm{or}(g)$. When $f$ is not an automorphism, then, the orientation of $f$ is only defined over its (invariant) automorphic domain. In the scalar case, the orientation can be interpreted as spring vs damper-like behaviour, in higher dimensions one can think of the map relating to a *flow* or not, see Remark II.4 below. At last we state the main tool of this section, which supplies us with an easy sufficient condition to assess $f \overset{t}{\sim} g$.

[2]For example, for $\mathcal{V} = (\mathbb{R}^n, +)$ linear maps are endomorphisms as they preserve the group-structure of $\mathbb{R}^n$. If these maps are invertible they are called *automorphisms*.

[3]Two topological spaces $\mathcal{X}$ and $\mathcal{Y}$ are *homeomorphic* when there exists a continuous bijective map $\varphi : \mathcal{X} \to \mathcal{Y}$ with a continuous inverse. Such a map $\varphi$ is called a *homeomorphism*.

**Theorem II.3** (Topological equivalence of asymptotically stable systems [17, Theorem 9.2 page 117] )**.** *Let $f(x) := Fx$ and $g(y) := Gy$ be asymptotically stable linear automorphisms on $\mathbb{R}^n$. Moreover, let $X(t)$ parametrize a path in $\mathsf{GL}(n,\mathbb{R})$, continuously depending on $t \in [0,1]$, such that $X(0) = F$ and $X(1) = G$, then, $f \overset{t}{\sim} g$.*

The key in Theorem II.3 is to demand that $F$ and $G$ are members of the same path-connected component of $\mathsf{GL}(n,\mathbb{R})$, hence the maps $f$ and $g$ have the same orientation.

**Remark II.4** (Orientation in practise)**.** Orientation might seem like an esoteric property, but it makes its appearance especially often when one discretizes a continuous-time problem. For instance, sampling any solution to $\dot{x} = Ax$ yields the time-one map $x \mapsto \exp(\mu A)x$ for some sampling step $\mu > 0$. It can be seen from $\det(\exp(A)) = \exp(\mathrm{Tr}(A))$ that this map is always orientation-preserving. It is known that this observation extends to non-linear systems [20], *e.g.,* the same holds (locally) for any Poincaré return map, which follows from the Liouville formula (*cf.* [18, Chapter 1]). This means that if one imposes a control law on these discretized maps which flips the orientation, then, this resulting map could never relate to some continuous flow.

### *B. Linear quadratic optimal control*

In this section we introduce the control problem at hand. Consider the deterministic linear discrete-time system

$$x \mapsto Ax + Bu =: \sigma(x,u), \quad x \in \mathbb{R}^n \tag{3}$$

where $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ compromise a stabilizable pair, that is, there exists a $K \in \mathbb{R}^{m\times n}$ such that $\rho(A + BK) < 1$. We will write this as $\sigma \in \Sigma$, for $\Sigma$ parametrized by the set of stabilizable pairs $(A,B)$. Then, for some triple $(Q,R,S) \in \mathcal{S}^n_{\succeq 0} \times \mathcal{S}^m_{\succ 0} \times \mathbb{R}^{n\times m}$ define the corresponding stage-cost $c : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ by

$$c(x,u) := \begin{pmatrix} x \\ u \end{pmatrix}^{\mathsf{T}} \begin{pmatrix} Q & S \\ S^{\mathsf{T}} & R \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix}. \tag{4}$$

When $S = 0$, we refer to the (stage-)cost as being *block-diagonal*. This diagonal form is well-understood and dominates the practical and theoretical literature. Now, following [21, Chapter 16], we can easily bring (4) to such a block-diagonal form. Specifically, by defining $v := R^{-1}S^{\mathsf{T}}x + u$, $Q' := Q - SR^{-1}S^{\mathsf{T}}$ and $A' := A - BR^{-1}S^{\mathsf{T}}$, we can, equivalently to (4) under (3), consider the stage-cost $c'(x,v) := x^{\mathsf{T}}Q'x + v^{\mathsf{T}}Rv$, under the time-one map $x \mapsto A'x + Bv$. This transformation allows for applying all celebrated block-diagonal tools.

In what follows we will assume (without loss of generality), that the input $u$ is linear state-feedback, that is, for some $K \in \mathbb{R}^{m\times n}$, $u := Kx$. Then, fixing some $\sigma \in \Sigma$, we define the cost function $J : \mathbb{R}^n \times \mathbb{R}^{m\times n} \to \mathbb{R} \cup \{\pm\infty\}$ by

$$\begin{aligned} J(x',K) :=\ & \textstyle\sum_{k=0}^{\infty} c(x_k, Kx_k), \\ \text{subject to }\ & x_{k+1} = \sigma(x_k, Kx_k)\ \forall k,\ x_0 = x'. \end{aligned} \tag{5}$$

To optimize this cost over $K$ and have a meaningful solution, assume the stage-cost is non-negative and that $(A,C)$ is a detectable pair for $C \in \mathbb{R}^{p\times n}$ defined by $C^{\mathsf{T}}C := Q$ with $\mathrm{rank}(Q) = \mathrm{rank}(Q')$. It is known that under the aforementioned conditions (*cf.* [21, Chapter 13-16]), $\arg\min_K J(x',K)$ is given by $K^\star = -(R + B^{\mathsf{T}}PB)^{-1}B^{\mathsf{T}}PA' - R^{-1}S^{\mathsf{T}}$, where $P \in \mathcal{S}^n_{\succeq 0}$ is the unique solution to the *Algebraic Riccati Equation* (ARE)

$$P = Q' + A'^{\mathsf{T}}PA' - A'^{\mathsf{T}}PB(R + B^{\mathsf{T}}PB)^{-1}B^{\mathsf{T}}PA', \tag{6}$$

such that the *optimal closed-loop* time-one map $x \mapsto \sigma(x, K^\star x) =: \sigma^\star(x)$ is asymptotically stable. With respect to (5), the aforementioned domain of $(Q,R,S)$ and the definition of $Q'$, we define the set of cost-matrices $\mathcal{C}(\sigma)$ by

$$\mathcal{C}(\sigma) := \left\{ (Q,R,S) : \begin{array}{c} Q - SR^{-1}S^{\mathsf{T}} \succeq 0, \\ \mathrm{rank}(Q) = \mathrm{rank}(Q'), \\ \exists\, C \in \mathbb{R}^{p\times n} : C^{\mathsf{T}}C = Q, \\ (A,C) \text{ detectable} \end{array} \right\}. \tag{7}$$

*1) Linear quadratic dynamic games:* One of the results of this note is that the qualitative behaviour of the whole *family* of block-diagonal Linear Quadratic Optimal Control problems is the same. To exemplify what we mean by this "*family*", we introduce a different, but analogous, block-diagonal cost function to (5). We introduce what is called a *two-player zero-sum dynamic game*, *e.g.,* see [22]. There, given some $\delta \in \mathbb{R}_{\geq 0}$, the stage-cost is defined by the function $g : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^d \to \mathbb{R}$

$$g(x,u,w) := x^{\mathsf{T}}Qx + u^{\mathsf{T}}Ru - \delta^{-1}w^{\mathsf{T}}w. \tag{8}$$

The variable $w$ will act as an *adversary*. Given some $D \in \mathbb{R}^{n\times d}$, let $\sigma_w(x,u) := \sigma(x,u) + Dw$ and again, without loss of generality, assume $w$ to be linear in $x$, that is $w := Lx$ for some $L \in \mathbb{R}^{d\times n}$. such that we can define the cost function $J : \mathbb{R}^n \times \mathbb{R}^{m\times n} \times \mathbb{R}^{d\times n} \to \mathbb{R} \cup \{\pm\infty\}$ by

$$\begin{aligned} J(x',K,L) :=\ & \textstyle\sum_{k=0}^{\infty} g(x_k, Kx_k, Lx_k), \\ \text{subject to }\ & x_{k+1} = \sigma_{Lx_k}(x_k, Kx_k)\ \forall k,\ x_0 = x'. \end{aligned} \tag{9}$$

Under conditions analogous to the ones from before (see [22, Chapter 3]), a solution to $\min_K \max_L J(x',K,L)$ exists and is given by the static gains $K^\star(\delta) := -R^{-1}B^{\mathsf{T}}P(\delta)\Lambda(\delta)^{-1}A$ and $L^\star(\delta) := \delta D^{\mathsf{T}}P(\delta)\Lambda(\delta)^{-1}A$. Here, the pair $(P(\delta), \Lambda(\delta))$ compromises a solution to the *Generalized Algebraic Riccati Equation* (GARE):

$$\begin{aligned} P &= Q + A^{\mathsf{T}}P\Lambda^{-1}A, \\ \Lambda &= \left(I_n + \left(BR^{-1}B^{\mathsf{T}} - \delta DD^{\mathsf{T}}\right)P\right). \end{aligned} \tag{10}$$

The parameter $\delta$ relates to how much adversarial action we allow for. Crossing what is called the "*breakdown-point*" $\bar{\delta}$ means it is "*affordable*" for the adversary to destabilize the system and hence this scenario is avoided by selecting $\delta \in [0, \bar{\delta})$ (see [22], [23]). Moreover, it can be shown that the closed-loop system matrix is asymptotically stable and can be written as $\Lambda(\delta)^{-1}A$. This observation is the key in showing topological equivalence in the next section.

Summarizing, we parametrize a Linear Quadratic (LQ) Optimal Control (OC) problem by the pair $(\sigma, J)$, where one seeks a sequence of inputs to the linear dynamical system $\sigma$ such that the quadratic cost function $J$, subject to $\sigma$, is minimized. Then, we will be interested in understanding to which topological class the corresponding optimal closed-loop systems $x \mapsto \sigma^\star(x)$ belong.

## III. Topological equivalence in linear quadratic optimal control

The main result of this note can be stated as follows: given any two closed-loop maps $f, g$ resulting from LQ optimal control problems with block-diagonal cost, then $f$ and $g$ are topologically equivalent. This means that the *qualitative* behaviour of these controlled systems is invariant under "*tuning*" of the cost. To proceed, we introduce a *orientation-dependent* version of the set in (7). Given a $\sigma \in \Sigma$ such that $A \in \mathsf{GL}(n,\mathbb{R})$ then define $\mathcal{C}^{(i)}(\sigma)$, $(i) \in \{+,-\}$ by

$$\mathcal{C}^{(i)}(\sigma) := \left\{ (Q,R,S) \in \mathcal{C}(\sigma) : \begin{matrix} A \in \mathsf{GL}^{(i)}(n,\mathbb{R}), \\ A' \in \mathsf{GL}^{(i)}(n,\mathbb{R}) \end{matrix} \right\}. \quad (11)$$

Here, $\mathsf{GL}^{(i)}(n,\mathbb{R})$ relates to either $\mathsf{GL}^{+}(n,\mathbb{R})$ or $\mathsf{GL}^{-}(n,\mathbb{R})$. Now we can state the main result.

**Theorem III.1** (Topological equivalence in LQ regulation)**.** *Fix some $\sigma \in \Sigma$ with $A \in \mathsf{GL}^{(i)}(n,\mathbb{R})$, $(i) \in \{+,-\}$. Let $x \mapsto \sigma_1^\star(x)$ be the optimal LQ regulated closed-loop time-one map corresponding to an arbitrary triple $(Q_1,R_1,S_1) \in \mathcal{C}^{(i)}(\sigma)$, that is $x \mapsto (A+BK_1^\star)x = \sigma_1^\star(x)$ with $K_1^\star$ the minimizing argument in* (5)*. Analogously, define $\sigma_2^\star$ for some arbitrary triple $(Q_2,R_2,S_2) \in \mathcal{C}^{(i)}(\sigma)$. Then, $\sigma_1^\star \overset{t}{\sim} \sigma_2^\star$.*

*Proof.* Since the cost-matrices are elements of $\mathcal{C}^{(i)}(\sigma)$ we use the transformations as set forth in Section II-B. Then, it is known (see [21, Chapter 12]) that the closed-loop system matrices can be written as $(I_n + BR_j^{-1}B^\mathsf{T}P_j)^{-1}A_j' =: \Lambda_j^{-1}A_j'$ for $P_j \in \mathbb{S}^n_{\succeq 0}$ the solution to the algebraic Riccati equation (6) under $(Q_j,R_j,S_j)$ $j \in \{1,2\}$. Now we claim that $\Lambda_j \in \mathsf{GL}^+(n,\mathbb{R})$. Let $N := BR^{-1}B^\mathsf{T}$ and $M := P_j$, which are both symmetric positive semidefinite. Then observe that given some eigenpair $(\lambda, v)$ such that $NMv = \lambda v$, multiplying from the left with $v^\mathsf{T}M^\mathsf{T}$ implies that $v^\mathsf{T}Mv\lambda = v^\mathsf{T}M^\mathsf{T}NMv \geq 0$ and hence, by construction of $M$ and $N$, that $\lambda \geq 0$. Therefore, the eigenvalues of $\Lambda_j$ are all strictly positive such that $\det(\Lambda_j) > 0$. Since an application of such a $\Lambda_j$ does not alter the membership of $A'$ to $\mathsf{GL}^{(i)}(n,\mathbb{R})$, *i.e.,* $\mathsf{GL}^+\mathsf{GL}^{(i)} = \mathsf{GL}^{(i)}$, we can directly appeal to Theorem II.3 and conclude the proof. □

We see from Theorem III.1 that if $(Q,R,0) \in \mathcal{C}(\sigma)$, then, since $A = A'$ we have $(Q,R,0) \in \mathcal{C}^{(i)}(\sigma)$ and indeed we see that *all* block-diagonal problems result in closed-loop maps being topologically equivalent. In particular, if $\rho(A) < 1$, then $Ax \overset{t}{\sim} (A+BK^\star)x$. Moreover, when $A \in \mathsf{GL}^+(n,\mathbb{R})$, then, block-diagonal LQ feedback leaves the group-structure intact, *i.e.,* $(A+BK^\star) \in \mathsf{GL}^+(n,\mathbb{R})$. Moreover, if $A$ is singular, we can still apply the idea of Theorem III.1, yet we need to restrict our discussion to the automorphic part of $\sigma^\star$, which is remarkably simple since $\ker(A)$ is preserved under block-diagonal LQ feedback (since the closed-loop matrix is of the form $\Lambda^{-1}A$ or see [24, Lemma 3.4]). When we introduce a non-zero $S$, however, the kernel of $A$ and the optimally LQ controlled closed-loop system matrix $\Lambda^{-1}A'$ do not necessarily match anymore since $\ker(A)$ and $\ker(A - BR^{-1}S^\mathsf{T})$ can be different.

The form of Theorem III.1 is chosen — since especially in the block-diagonal case — it captures the central message: without constructing explicit LQR solutions one can easily assess *a priori* if some closed-loop maps will be topologically equivalent. However, in line with Theorem II.3, the statement could be made more general and *direct* by demanding that the closed-loop maps are stable and have the same orientation, *i.e.,* $\mathrm{or}(\Lambda_1^{-1}A_1') = \mathrm{or}(\Lambda_2^{-1}A_2')$. We refrain from doing so and instead construct another indirect characterization of these distinct topological classes.

When $A \in \mathsf{GL}(n,\mathbb{R})$, then, a minimizing solution to the standard LQR cost (5) can be characterized via a Symplectic matrix. Particularly, define $\Omega \in \mathbb{R}^{2n\times 2n}$ by

$$\Omega = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}.$$

Then, define the real *Symplectic group* by $\mathsf{Sp}(2n,\mathbb{R}) := \{M \in \mathbb{R}^{2n\times 2n} : M^\mathsf{T}\Omega M = \Omega\}$. Moreover, we speak of a subspace $\mathcal{Y}$ being $M$-invariant, when $M\mathcal{Y} \subseteq \mathcal{Y}$. Next, define $M \in \mathsf{Sp}(2n,\mathbb{R})$ by

$$M := \begin{bmatrix} A' + BR^{-1}B^\mathsf{T}A'^{-\mathsf{T}}Q' & -BR^{-1}B^\mathsf{T}A'^{-\mathsf{T}} \\ -A'^{-\mathsf{T}}Q' & A'^{-\mathsf{T}} \end{bmatrix}. \quad (12)$$

A celebrated result — as for example communicated for an even more general setting in [25] — is that eigenspaces of $M$ in (12) directly map to solutions of (6) and in fact, the spectrum of $M$ relates directly to the spectrum of the optimal LQ regulated time-one map. Better yet, the relation between $M$ and $\Lambda^{-1}A'$ is well-understood. Now, assume that the triple $(Q,R,S)$ is parametrized by some scalar $\gamma$, that is, let $A'(\gamma) := A - BR(\gamma)^{-1}S(\gamma)^\mathsf{T}$, $Q'(\gamma) := Q(\gamma) - S(\gamma)R(\gamma)^{-1}S(\gamma)^\mathsf{T}$ and define $M(\gamma) \in \mathsf{Sp}(2n,\mathbb{R})$ accordingly. Then, loosely speaking, it turns out that when $M(\gamma)$ is a continuous curve in $\mathsf{Sp}(2n,\mathbb{R})$, then, all closed-loop maps it parametrizes (see Section II-B) are topologically equivalent. We formalize this in a Corollary to Theorem III.1:

**Corollary III.2** (Topological equivalence via the Symplectic Group)**.** *Fix some $\sigma \in \Sigma$ with a $A \in \mathsf{GL}(n,\mathbb{R})$ and let $\gamma \in [0,1]$ parametrize a curve $(Q,R,S)(\gamma) \subset \mathcal{C}(\sigma)$ such that both $(Q,R,S)(0)$ and $(Q,R,S)(1)$ correspond to feasible LQR problems with optimal closed-loop maps $\sigma^\star(x)(0)$ and $\sigma^\star(x)(1)$. Then, $\sigma^\star(0) \overset{t}{\sim} \sigma^\star(1)$ if there exists a continuous path $[0,1] \mapsto M[0,1] \subset \mathsf{Sp}(2n,\mathbb{R})$ from $M(0)$ to $M(1)$.*

*Proof.* Since $\mathsf{Sp}(2n,\mathbb{R}) \subset \mathsf{SL}(2n,\mathbb{R})$ and we can continuously deform $M(0)$ into $M(1)$ this must mean we do not drop rank along the path $M(\gamma)$, $\gamma \in [0,1]$. Moreover, we know that for any $M \in \mathsf{Sp}(2n,\mathbb{R})$ $\mu \in \lambda(M) \Rightarrow 1/\mu \in \lambda(M)$. Also, for any feasible LQR problem leading to (12) it is known that $1 \notin |\lambda(M)|$, hence, when one constructs the Jordan normal form related to such a $M$, there are $n$-dimensional $M$-invariant *stable* ($s$) and *unstable* ($u$) subspaces, that is $M = XJX^{-1}$ with $X = [X^s \ X^u]$, $J = \mathrm{diag}(J^s, J^u)$. In fact, it can be shown that $\lambda(J^s) = \lambda(\Lambda^{-1}A')$, where $\Lambda^{-1}A'$ is the optimal LQ regulated closed-loop system matrix from Section II-B. See [25] and references therein for more on the aforementioned results. The prior discussion implies that if $0 \notin \lambda(M(\gamma))\ \forall \gamma \in [0,1]$, then, $0 \notin \lambda(J^s(\gamma))$ and as such $0 \notin \lambda(\Lambda(\gamma)^{-1}A'(\gamma))$. This however means that $A'(\gamma) \in \mathsf{GL}^{(i)}(n,\mathbb{R})\ \forall\gamma \in [0,1]$, thereby, the result follows after an application of Theorem II.3. $\square$

So, although $\mathsf{Sp}(2n,\mathbb{R})$ is connected, by varying the triple $(Q,R,S) \in \mathcal{C}(\sigma)$ we effectively generate disjoint connected sets of matrices $M \in \mathsf{Sp}(2n,\mathbb{R})$ in (12) corresponding to distinct topological classes of closed-loop maps they generate. Since this result provides the link with the far more general *Maximum Principle*, the hope is that similar constructions are possible for different Hamiltonians.

Now, we are not just interested in the OC problem related to (5), but into the whole "*family of LQ problems*". To that end, we give one example and use Section II-B.1 to extend Theorem III.1.

**Corollary III.3** (Topological equivalence in dynamic games)**.** *Fix $\sigma \in \Sigma$, let $A \in \mathsf{GL}^{(i)}(n,\mathbb{R})$ and set $D := I_n$ in* (9)-(10)*. Moreover, consider $J$ as in* (9) *for some pair $(Q,R)$ being such that for any $\delta \in [0,\bar{\delta})$ the extremizers in $\min_{K\in\mathbb{R}^{m\times n}} \max_{L \in \mathbb{R}^{d\times n}} J(x',K,L)$ denoted by $K^\star(\delta)$ and $L^\star(\delta)$, exist. Then, the "nominal"-, "robust"- and "worst-case robust" optimal closed-loop maps given by $f(x) := \big(A + BK^\star(\delta)\big)x|_{\delta=0}$, $g(x) := \big(A+BK^\star(\delta)\big)x|_{\delta\in(0,\bar{\delta})}$, $h(x) := \big(A + BK^\star(\delta) + L^\star(\delta)\big)x|_{\delta\in(0,\bar{\delta})}$, respectively, are topologically equivalent.*

*Proof.* Let the pair $(P(\delta),\Lambda(\delta))$ correspond to a solution to (10). Recall from for example [22, Chapter 3] that $f(x) = \Lambda^{-1}(\delta)Ax|_{\delta=0}$, $g(x) = \big(I_n - \delta P(\delta)\big)\Lambda^{-1}(\delta)Ax|_{\delta\in(0,\bar{\delta})}$ with $(\delta^{-1}I_n - P(\delta)) \succ 0$ and $h(x) = \Lambda^{-1}(\delta)Ax|_{\delta\in(0,\bar{\delta})}$. Then, all that we need to show is that $\Lambda(\delta)|_{(0,\bar{\delta})} \in \mathsf{GL}^+(n,\mathbb{R})$. It follows from Theorem III.1 that $\lim_{\delta\downarrow 0}\Lambda(\delta) = (I_n + BR^{-1}B^\mathsf{T}P) \in \mathsf{GL}^+(n,\mathbb{R})$. Since $\mathsf{GL}(n,\mathbb{R})$ has two connected components, $\Lambda(\delta)$ is continuous in $\delta$ (this can be shown as in [26]) and starts in $\mathsf{GL}^+(n,\mathbb{R})$, it must remain in that group. This concludes the proof. $\square$

Corollary III.3 indicates that the adversaries these dynamic games hedge against are somewhat *natural*.

**Remark III.4** (Beyond standard LQR)**.** Corollary III.3 shows that Theorem III.1 is not limited to the "*standard*" LQR problem. Hence, we would like to point to related problems, displaying similar, if not equivalent, structure. When considering an exponential utility function in (5) subject to a linear Gaussian system — which is called the LEQR problem — then, its optimal policy coincides with that of a dynamic game [22], [23], [27]. Similar algebraic structures are also seen in distributionally robust control and estimation [28], [29]. Note that in the stochastic case, the topological equivalence is with respect to the closed-loop mean state processes. Also, to be able to use Theorem II.3 in a discounted setting, stability must be explicitly verified.

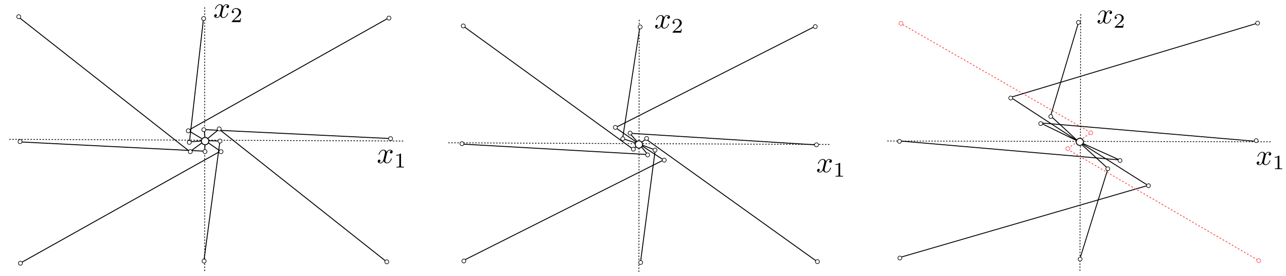
We can conclude, however, without a formal proof, that given any two optimal closed-loop time-one maps resulting from any two block-diagonal LQ OC problems, they are topologically equivalent. The crux is that all these optimal closed-loop maps are of the form $x \mapsto \Lambda^{-1}Ax$ for some $\Lambda \in \mathsf{GL}^+(n,\mathbb{R})$. Now, if the cost is not block-diagonal, the cross-terms will determine, for the better or worse, to which topological class the closed-loop map belongs.

This note also showed again the importance of correctly identifying $\ker(A)$ and $\mathrm{or}(A)$. If some estimate of $A$, say $\widehat{A}$, satisfies $\mathrm{or}(A) \neq \mathrm{or}(\widehat{A})$, then, for standard (block-diagonal) LQR, no matter the tuning, the simulated and real behaviour will always be structurally different. This problem was recently addressed in [30] where the authors project the estimated system matrix to the set of asymptotically stable matrices and characterize the probability of identifying the correct orientation. Correctly identifying the orientation of the drift term $A$ also relates to the recent work [31] were an adaptive control law is designed to deal with a map of the form $x \mapsto iAx$, $x \in \mathbb{R}^n$ for some unknown (adversarially chosen) $i \in \{-1,1\}$. As hinted at before, one can also apply these ideas in the stabilization of periodic orbits, for example, in robot locomotion. This work provides new motivation for why one could select a local LQ regulator over a more traditional PID regulator.

## IV. More on "tuning"

In the vast majority of work on LQ optimal control the stage-cost (4) is *diagonal* (*cf.* [2], [32]). However, we will emphasize that one should not underestimate the applicability of $S$. One successful example is presented in [33], where the authors exploit the cross-term in the cost with the purpose of penalizing subsequent input deviations.

Moreover, assume we have access to data sampled from a closed-loop linear system. We might ask, is there a LQ Optimal Control (OC) problem which gives rise to this system? This is a question of *inverse optimal control* (IOC). It was remarked in [34] that any linear feedback gain $K' \in \mathbb{R}^{m\times n}$ corresponds to some LQ OC problem, *e.g.,* define the stage-cost $c(x,Kx) := x^\mathsf{T}(K-K')^\mathsf{T}(K-K')x$. So, from an IOC point of view, the matrix $S$ seems not the most interesting, everything is possible. Looking at this from the tuning point of view, by excluding $S$, you are restricting the behaviour of the closed-loop system to maps with at least the same orientation as the automorphic part of $Ax$, *e.g.*, in the scalar case, by changing the pair $(Q,R)$, one cannot go from *spring*- to *damper*-like behaviour. Therefore, we propose that if one wants to tune, if a change in behaviour is desired, change $S$. It is imperative to remark that when $A \in \mathsf{GL}(n,\mathbb{R})$, then, the LQ optimal closed-loop systems are *structurally* stable with respect to a perturbation in the tuple $(A,C,R,S)$. At last, we briefly visualize the effect of $S$.




(a) ($\varepsilon = 0$) A (discrete) stable counter-clockwise spiral with $\mathrm{or}(\sigma^\star) = 1$.

(b) ($\varepsilon = 0.5$) Due to an increase in $\varepsilon$ the spiral starts to shear, yet $\mathrm{or}(\sigma^\star) = 1$.

(c) ($\varepsilon = 2$) For $\varepsilon > 1$ $\mathrm{or}(\sigma^\star) = -1$ and the spiralling behaviour deteriorates.

Fig. 2: Given the parameters from Example IV.1, we show a few closed-loop trajectories as a function of $S(\varepsilon)$.

**Example IV.1** (Structural Tuning)**.** Consider the general LQR problem from Section II-B parametrized by $B = I_2$, $R = I_2$, $Q = 10 \cdot I_2$ and

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad S(\varepsilon) = \begin{pmatrix} \varepsilon & 0 \\ 0 & -\varepsilon \end{pmatrix},$$

for some $\varepsilon \in \mathbb{R}_{\geq 0}$. We see that for all $\varepsilon < 1$, $S(\varepsilon) \in \mathcal{C}^+(\sigma)$, *i.e.,* $\det(A - S(\varepsilon)) = -\varepsilon^2 + 1$. To illustrate the structural change, we vary $\varepsilon$ from 0 and 2, construct $K^\star$ accordingly and show a few closed-loop, that is $x \mapsto (A + BK^\star)x$, trajectories in Figure 2. Indeed, once $S$ leaves $\mathcal{C}^+(\sigma)$, the behaviour changes structurally (*cf.* Figure 2c).

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