Small errors in random zeroth-order optimization are imaginary

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Based on: "*Small errors in random zeroth-order optimization are imaginary*"
arXiv: https://arxiv.org/abs/2103.05478.
by **Wouter Jongeneel** (EPFL), Man-Chung Yue (HKU) and Daniel Kuhn (EPFL).

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First-order optimization 101

For $f \quad C^1(X \quad \mathrm{R}^n; \mathrm{R})$, how to find

$$x \quad \operatorname{argmin}_x \quad_X f(x)\,?$$

[1] Popularity measure: in the last year (May 2022 - May 2023), searching for "SGD" was on average just a factor 1/25 as popular a searching for "Covid" (worldwide) https://trends.google.com/trends/explore?q=SGD,Covid.

[2] Still a very active research topic,
see https://www.quantamagazine.org/risky-giant-steps-can-solve-optimization-problems-faster-20230811/.

[3] Nesterov 2003, § 2.1.5.

First-order optimization 101

For $f \in C^1(X \subseteq \mathrm{R}^n; \mathrm{R})$, how to find

$$x^* \in \operatorname{argmin}_{x \in X} f(x)\,?$$

Common[1] approach: ***gradient*** *descent*

$$x_{k+1} = x_k - \mu_k \nabla f(x_k), \quad k = 1, 2, \ldots \tag{1}$$

Let f be convex with a L-Lipschitz **gradient**, *i.e.*,

$$\|\nabla f(x) - \nabla f(y)\|_2 \le L \|x - y\|_2, \qquad x, y \in X,$$

then, for[2] $\mu_k = 1/L$ and $x_1, x_2, \ldots, x_K$ generated by (1) one obtains[3]

$$f(x_K) - f(x^*) \le O\left(\frac{L \cdot \|x_1 - x^*\|_2^2}{K}\right).$$

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If a gradient exists, does it mean we always *have* a gradient?

Example: DE constrained problems

Energy efficiency of transportation systems becomes increasingly important; must be optimized[4]. Good news: regularity is understood/studied.

Let $f(x)$ represent *aerodynamic performance* for x a set of *design parameters*, do we have an expression for $f(x)$?

[4] Images from: `https://predatorcycling.com/`, `https://www.3ds.com/` and `https://www.youtube.com/watch?v=FGmYpo-gkpU&ab_channel=EdwinLinders`.

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◦ Idea: we *can* **evaluate** $f(x)$ for some design choice x, *i.e.*, by simulation, and subsequently use $x_1, x_2, \ldots, f(x_1), f(x_2), \ldots,$ (**zeroth-order information**) for optimization.

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Zeroth-order optimization

Obtain (approximate)

$$x \qquad \operatorname{argmin}_x \; {}_X \, f(x)$$

via function evaluations $f(x_1), f(x_2), \ldots, f(x_K)$ for some set of *selected* points $x_1, x_2, \ldots, x_K$. (For simplicity, we omit noise for now.)

[5] For references, consult the recent survey articles: Larson, Menickelly, and Wild 2019; Liu et al. 2020.

[6] See the books by Conn, Scheinberg, and Vicente 2009 and Audet and Hare 2017.

[7] Kiefer and Wolfowitz 1952; Nemirovsky and Yudin 1983; Flaxman, Kalai, and McMahan 2004; Spall 2005; Nesterov and Spokoiny 2017.

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Two common paths[5]:

(i) *Approximate a model*: construct a local model of f, optimize using that model, *e.g.*, using a trust region method[6].

(ii) ***Approximate an algorithm***: *e.g.*, approximate f directly and apply some form of gradient descent[7].

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A gradient-based approach

For any smooth $f : \mathrm{R} \to \mathrm{R}$

$$\partial_x f(x) = \frac{f(x+\delta) - f(x)}{\delta} + O(\delta).$$

[8] d'Aspremont 2008; Devolder, Glineur, and Nesterov 2014.

A gradient-based approach

For any smooth $f : \mathrm{R} \to \mathrm{R}$

$$\partial_x f(x) = \frac{f(x+\delta) - f(x)}{\delta} + O(\delta).$$

Then, run ***inexact*** ($\delta > 0$ fixed) gradient descent

$$x_{k+1} = x_k - \mu_k \frac{f(x_k+\delta) - f(x_k)}{\delta}.$$

◦ When does $f(x_k) \to f(x^*)$?

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A gradient-based approach

For any smooth $f : \mathbb{R} \to \mathbb{R}$

$$@_x f(x) = \frac{f(x+\ ) - f(x)}{} + O(\ ):$$

Then, run inexact (> 0 xed) gradient descent

$$x_{k+1} = x_k \quad {}_k \frac{f(x_k+\ ) - f(x_k)}{}:$$

When does $f(x_k) \to f(x^?)$?

For xed > 0, a bias prevails, $f(x_k) \to f(x^?) + O(\ )$[8], e.g., for $f(x) = x^2$ we e ectively compute the gradient of $f(x) + x$, shifting $x^? = 0$ to $\frac{1}{2}$.
Similarly, for $f \in C^1(\mathbb{R}^n; \mathbb{R})$, one should not naïvely use

$$\sum_{i=1}^{n} \frac{f(x+\ e_i) - f(x)}{} e_i \quad \text{for} \quad (e_1; e_2; \dots; e_n) = I_n:$$

[8] d'Aspremont 2008; Devolder, Glineur, and Nesterov 2014.

A gradient-based approach cont.

(i) For appropriate (adaptive) > 0, apply line-search[9] using

$$\sum_{i=1}^{n} \frac{f(x + b_i) - f(x)}{b_i} b_i \approx \nabla f(x); \quad \text{for} \quad \det(b_1; b_2; \ldots; b_n) \neq 0:$$

[9] Berahas, Cao, and Scheinberg 2021.

[10] Randomization can be optimal Duchi et al. 2015, but no uniformly superior method exists yet randomized finite difference schemes can be implemented to be n times cheaper [than deterministic finite difference]; but an algorithm based on them has to take at least n times more steps. Scheinberg 2022, see also Berahas et al. 2022.

A gradient-based approach cont.

(i) For appropriate (adaptive) > 0, apply line-search[9] using

$$\sum_{i=1}^{n} \frac{f(x + b_i) - f(x)}{} b_i \quad r \quad f(x); \quad \text{for} \quad \det(b_1; b_2; : : : ; b_n) \neq 0 :$$

(ii) Suppose we nd a random variable $2 R^n$ such that

$$E \left[\frac{f(x +) - f(x)}{} \right] \quad r \quad f(x):$$

Consider the randomized algorithm

$$x_{k+1} = x_k \quad _k \frac{f(x_k +) - f(x_k)}{} ; \qquad :$$

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(!) Active topic of research[10].

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Highly-in uential exercise by Nemirovski and Yudin

Let $f : R^n \to R$, Nemirovski and Yudin[11] consider: -smoothing

$$f(x) = E_{y \in B^n}[f(x + y)] = \mathrm{vol}(B^n)^{-1} \int_{B^n} f(x + y)dy; \quad (2a)$$

$$r f(x) = \frac{n}{-} E_{y \in S^{n-1}}[f(x + y)y] = \frac{n}{-} \int_{S^{n-1}} f(x + y)y \, (dy): \quad (2b)$$

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Natural single-point candidate to approximate @f:

$$g(x) = \frac{n}{-} f(x + y)y; \quad y \in S^{n-1}: \qquad (3a)$$

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Highly-in uential exercise by Nemirovski and Yudin

Let $f : R^n \to R$, Nemirovski and Yudin[11] consider: -smoothing

$$f_\mu(x) = E_{y \sim B^n}[f(x + \mu y)] = \mathrm{vol}(B^n)^{-1} \int_{B^n} f(x + \mu y)\,dy; \qquad (2a)$$

$$\nabla f_\mu(x) = \frac{n}{\mu} E_{y \sim S^{n-1}}[f(x + \mu y)\, y] = \frac{n}{\mu} \int_{S^{n-1}} f(x + \mu y)\, y\, \mu(dy): \qquad (2b)$$

Natural single-point candidate to approximate ∇f:

$$g_\mu(x) = \frac{n}{\mu} f(x + \mu y)\, y; \quad y \sim S^{n-1}: \qquad (3a)$$

Observation[12]: avoid high-variance for $\mu \to 0$ and give (3a) again the interpretation of a directional derivative and use a multi-point oracle like:

$$g_\mu^0(x) = \frac{n}{\mu} (f(x + \mu y) - f(x))\, y; \quad y \sim S^{n-1}: \qquad (3b)$$

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Early algorithmic analysis by Nesterov and Spokoiny

For $f : R^n \to R$ (locally convex), Gaussian smoothing[13]

$$f_\mu(x) = \frac{1}{\mu} \int f(x + \mu y) e^{-\frac{1}{2}\|y\|_2^2} dy \tag{4a}$$

$$r f_\mu(x) = \frac{1}{\mu} \int_{R^n} \frac{f(x + \mu y) - f(x - \mu y)}{2} e^{-\frac{1}{2}\|y\|_2^2} y dy \tag{4b}$$

with $\|r f_\mu - r f\|_\infty = O(n^2)$.

[13] Nesterov 2011; Nesterov and Spokoiny 2017.

Early algorithmic analysis by Nesterov and Spokoiny

For $f : \mathbb{R}^n \to \mathbb{R}$ (locally convex), Gaussian smoothing[13]

$$f_\gamma(x) = \frac{1}{\gamma} \int f(x + \gamma y) e^{-\frac{1}{2}\|y\|_2^2} dy \tag{4a}$$

$$\nabla f_\gamma(x) = \frac{1}{\gamma} \int_{\mathbb{R}^n} \frac{f(x + \gamma y) - f(x - \gamma y)}{2} e^{-\frac{1}{2}\|y\|_2^2} y dy \tag{4b}$$

with $\|\nabla f_\gamma - \nabla f\|_2 = O(n\gamma^2)$.

$$\text{Oracle (cd):} \quad g(x) = \frac{f(x + \gamma y) - f(x - \gamma y)}{2} y; \quad y \sim N(0; I_n)$$

with $E_{u \sim N(0;I_n)} \|g(x)\|_2^2 \le O(n^2\gamma^2 + n\|\nabla f(x)\|_2^2)$.

$$\text{Algorithm:} \quad x_{k+1} = x_k - \eta_k g_k(x_k); \quad \eta_k = O(1/nL):$$

$$\text{Performance:} \quad \text{for } \eta_k \to 0 \text{ and } x_K := 1/K \sum_{k=1}^{K} x_k$$

$$E[f(x_K)] - f(x^*) \le O\left(\frac{nL\|x_1 - x^*\|_2^2}{K}\right) = O(n) \quad \text{gradient descent}$$

[13] Nesterov 2011; Nesterov and Spokoiny 2017.

Numerical considerations

Analysis continued after 2011-2017, still, all common[14] oracles of the form

$$(\text{ nite di erence }): \quad \frac{f(x+\ y)\quad f(x)}{\quad}y = @_x f(x) + O(\ )$$

$$(\text{central di erence}): \quad \frac{f(x+\ y)\quad f(x\quad y)}{2}y = @_x f(x) + O(\ ^2);$$

$$::: = @_x f(x) + O(\ ^{p\ \ 1})$$

As such, many algorithms require $_k \quad O\ (1=k^q)$, with $q > 0$ for $k = 1; 2; : : : .$

[14] Hazan and Levy 2014; Duchi et al. 2015; Nesterov and Spokoiny 2017; Gasnikov et al. 2017; Shamir 2017; Akhavan, Pontil, and Tsybakov 2020; Lam, Li, and Zhang 2021; Novitskii and Gasnikov 2021.

[15] Generally, $_M = 2^{\ 52}\ \ 10^{\ 16}$.

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As such, many algorithms require $_k \quad O\ (1=k^q)$, with $q > 0$ for $k = 1; 2; : : : .$

However, can we practically select $_k\ !\ \ 0$ for $k\ !\ \ +1$?

For su ciently small $\ $, $f(x+\ y)\quad f(x)$ machine precision[15]
=) cancellation error, i.e., oracle output is nonsense.

Not that frequently discussed, does it matter?

[14] Hazan and Levy 2014; Duchi et al. 2015; Nesterov and Spokoiny 2017; Gasnikov et al. 2017; Shamir 2017; Akhavan, Pontil, and Tsybakov 2020; Lam, Li, and Zhang 2021; Novitskii and Gasnikov 2021.

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Intermezzo: a beautiful insight from complex analysis

As pioneered in the 60s[16], let $f : \mathbb{R} \to \mathbb{R}$ be real analytic (C^ω) and consider

$$f(x + i\varepsilon) = f(x) + \varepsilon \partial_x f(x) - \frac{1}{2}\varepsilon^2 \partial_x^2 f(x) - \frac{1}{6} i\varepsilon^3 \partial_x^3 f(x) + O(\varepsilon^4); \quad i^2 = -1:$$

such that (for $z \in \mathbb{C}$, $z = <(z) + =(z)$):

$$=(f(x + i\varepsilon)) = \varepsilon \partial_x f(x) - \frac{1}{6}\varepsilon^3 \partial_x^3 f(x) + O(\varepsilon^5)$$

[16] Lyness and Moler 1967; Squire and Trapp 1998; Martins, Sturdza, and Alonso 2003; Abreu et al. 2018.

[17] A value of $\varepsilon = 10^{-100}$ (!) is successfully used in National Physical Laboratory software Cox and Harris 2004, Page 44.

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As pioneered in the 60s[16], let $f : \mathbb{R} \to \mathbb{R}$ be real analytic (C^ω) and consider

$$f(x + ih) = f(x) + h\partial_x f(x) - \frac{1}{2}h^2\partial_x^2 f(x) - \frac{1}{6}ih^3\partial_x^3 f(x) + O(h^4); \quad i^2 = -1:$$

such that (for $z \in \mathbb{C}$, $z = \Re(z) + i\Im(z)$):

$$\Im(f(x + ih)) = h\partial_x f(x) - \frac{1}{6}h^3\partial_x^3 f(x) + O(h^5)$$

and thus

$$\partial_x f(x) = \frac{\Im f(x + ih)}{h} + O(h^2); \quad f(x) = \Re(f(x + ih)) + O(h^2):$$

Hence, consider using

$$\frac{\Im f(x + ih)}{h} \to \partial_x f(x):$$

Cancellation errors are impossible[17]. Again, does it matter?

[16] Lyness and Moler 1967; Squire and Trapp 1998; Martins, Sturdza, and Alonso 2003; Abreu et al. 2018.

[17] A value of $h = 10^{-100}$ (!) is successfully used in National Physical Laboratory software Cox and Harris 2004, Page 44.

Numerical considerations cont.: an example

For $f(x) = x^3$, approximate $r\ f(x)$ at $x \in f\ \ 1; 0; 10g$ using

$$\text{(forward di erence):} \quad f_{fd}(x; \ ) = \frac{f(x + \ ) \quad f(x)}{\ }; \tag{5a}$$

$$\text{(central di erence):} \quad f_{cd}(x; \ ) = \frac{f(x + \ ) \quad f(x \quad)}{2}; \tag{5b}$$

$$\text{(complex-step):} \quad f_{cs}(x; \ ) = \frac{= (f(x + i \ ))}{\ } \tag{5c}$$

and compare the error for $\ \# 0$:

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$$\text{(complex-step):} \quad f_{cs}(x; \ ) = \frac{= (f(x + i \ ))}{\ } \tag{5c}$$

and compare the error for $\ \neq 0$:

(a) $x = \ \ 1$ (b) $x = 0$ (c) $x = 10$

Failures well before $\ _M$, so, it does matter.

On the necessity of leaving R

Although single-point estimators exist[18], variance blows up for $\neq 0$. Is this complex-lifting business really needed? Is there not a real analogue of

$$\partial_x f(x) = \frac{=(f(x + i\varepsilon))}{\varepsilon} + O(\varepsilon^2)? \tag{6}$$

[18] Flaxman, Kalai, and McMahan 2004.

[19] Jongeneel 2021.

On the necessity of leaving $\mathbb{R}$

Although single-point estimators exist[18], variance blows up for $\epsilon \to 0$. Is this complex-lifting business really needed? Is there not a real analogue of

$$\partial_x f(x) = \frac{\Im(f(x + i\epsilon))}{\epsilon} + O(\epsilon^2)? \tag{6}$$

Partial answer[19]: no.

Consider some non-empty open, convex set $D \subseteq \mathbb{R}^n$ then, there does not exist a continuous map $G : \mathbb{R} \to \mathbb{R}$ such that for all real-analytic functions $f : D \to \mathbb{R}$

$$G(f(x + \epsilon y)) = \langle \nabla f(x), y \rangle + o(\epsilon) \quad \forall x \in D, \epsilon > 0, y \in S^{n-1}. \tag{7}$$

not surprising, but provides motivation.

[18] Flaxman, Kalai, and McMahan 2004.

[19] Jongeneel 2021.

Comment on Algorithmic Differentiation (AD)

Why bother with approximations?

[20] The Deep Learning Toolbox in MATLAB and AD tools in Julia (See Bezanson et al. 2017; Revels, Lubin, and Papamarkou 2016; Innes 2018; Moses and Churavy 2020), e.g., ForwardDiff.jl, Zygote.jl and Enzyme.jl or in Python, e.g., JAX Bradbury et al. 2018

Comment on Algorithmic Di erentiation (AD)

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Dual numbers: $a + b\varepsilon$ with $a, b \in \mathbb{R}$ and $\varepsilon \neq 0$, yet, $\varepsilon^2 = 0$, i.e., elements of the quotient ring $\mathbb{R}[\varepsilon]/\varepsilon^2$, not a field =) , e.g., $\varepsilon^2/\varepsilon$ and $\sqrt{\varepsilon^2}$ not defined.

C is an algebraically closed field.

AD: for $f : \mathbb{R} \to \mathbb{R}$ is sufficiently regular, e.g., $f \in C^\omega(\mathbb{R})$, then,
$f(x + \varepsilon) = f(x) + \partial_x f(x)\varepsilon$, i.e., $f(x + \varepsilon)$ provides us with the pair $(f(x), \partial_x f(x))$.

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Consider $\partial_x f(x)|_{x=0}$ for the following C^ω functions:

$$f(x) = x/x; \quad f(x) = \sin(x)/x; \quad f(x) = e^{\sqrt{x^2}-2}:$$

No free lunch: most populair AD tools[20] evaluate to NaN whereas the complex-step derivative correctly approximates $\partial_x f(x)|_{x=0} = 0$.
Theoretical solution: Levi-Civita field $\sum_{q \in Q} a_q \varepsilon^q$ with $a_q \in \mathbb{R}\ \forall q \in Q$ (inf. dim).

[20] The Deep Learning Toolbox in MATLAB and AD tools in Julia (See Bezanson et al. 2017; Revels, Lubin, and Papamarkou 2016; Innes 2018; Moses and Churavy 2020), e.g., ForwardDiff.jl , Zygote.jl and Enzyme.jl or in Python, e.g., JAX Bradbury et al. 2018

A solution: the complex-step oracle [22] (exercise)

Let $f \in C^{!}(X \quad R^{n}; R)$, using Cauchy-Riemann/Stokes show that:

$$f\ (x) = E_{y \quad B^{n}}\ [< (f\ (x + i\ y\ ))]$$

$$r\ f\ (x) = \frac{n}{\quad}\ E_{y \quad S^{n \quad 1}}\ [= (f\ (x + i\ y\ ))\ y]$$

with $kr\ f \quad r \quad f\ k_{2} \quad O\ (n^{\ 2})$.

[21] The paper provides similar results for strong-convex and non-convex functions. This approach recently surfaced in the optimization community Nikolovski and Stojkovska 2018; Hare and Srivastava 2023 with the rst complete deterministic non-asymptotic analysis appearing in Jongeneel, Yue, and Kuhn 2021. The rst applications of the complex-step derivative to Reinforcement Learning appeared in Wang and Spall 2021; Wang, Zhu, and Spall 2021.

[22] Jongeneel, Yue, and Kuhn 2021.

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with $kr\ f\quad r\quad f\ k_2\quad O\ (n^{\ 2})$.

Oracle (cs): $g\ (x) = \frac{n}{\quad} = (f\ (x + i\ y\ ))\ y; \quad y \quad S^{n\ 1}$:

with $E_{u\ S^{n\ 1}}\ kg\ (x)k_2^2 \quad O\ (n^{2\ 4} + n^{2\ 2}kr\ f\ (x)k_2 + nkr\ f\ (x)k_2^2)$.

Algorithm : $x_{k+1} = x_k \quad {}_k g_{\ k}(x_k); \quad {}_k = O\ (1=nL\ )$

Performance: for f convex ${}_k = O(1=k)$ and $x_K := 1 =K \sum_{k=1}^{K} x_k$

$$E[f\ (x_K\ )]\quad f\ (x^{?})\quad O\quad \frac{n\quad L\quad kx_1\quad x^{?}k_2^2}{K}\quad = O(n)$$ gradient descent[21]:

[21] The paper provides similar results for strong-convex and non-convex functions. This approach recently surfaced in the optimization community Nikolovski and Stojkovska 2018; Hare and Srivastava 2023 with the rst complete deterministic non-asymptotic analysis appearing in Jongeneel, Yue, and Kuhn 2021. The rst applications of the complex-step derivative to Reinforcement Learning appeared in Wang and Spall 2021; Wang, Zhu, and Spall 2021.

[22] Jongeneel, Yue, and Kuhn 2021.

Solution:

$f \in C^{!}$ convex $\not\Longrightarrow$ f convex, e.g.,

for $f(x) = x^4$ we have $<(f(x + i y)) = x^4 - 6x^2(y)^2 + (y)^4$.

Hence, look beyond typical convex proofs[23].

[23] We frequently appeal to Schmidt, Roux, and Bach 2011, Lem. 1.

[24] Although the general form is largely due to Cartan (Élie).

[25] $C^{!}$ is sufficient, but not necessary.

Solution:

$f \in C^{\omega}$ convex $\not\Longrightarrow$ f convex, e.g.,
for $f(x) = x^4$ we have $\Re(f(x+iy)) = x^4 - 6x^2(y)^2 + (y)^4$.
Hence, look beyond typical convex proofs[23].

Cauchy, Riemann and Stokes[24] meet:
for[25] $f \in C^{\omega}(\mathbb{R}^n;\mathbb{R})$ with $f(x+iy) = u(x;y) + iv(x;y)$, then
$\partial_{x_i} u = \partial_{y_i} v$; $\partial_{y_i} u = -\partial_{x_i} v$ $\forall i \in [n]$ (CR)
and for Ω orientable we have that $\int_{\Omega} d\omega = \int_{\partial\Omega} \omega$ (Stokes), implication: the divergence theorem $\int_{\Omega} \operatorname{div}(X)\,d\mathrm{vol} = \int_{\partial\Omega} \langle X, N\rangle\, d\mathrm{vol}_{\partial\Omega}$. Hence:

$$
\begin{aligned}
\nabla f(x) &\overset{(\text{def.; DCT})}{=} \operatorname{vol}(B^n)^{-1} \int_{B^n} \nabla_x \Re(f(x+iy))\,dy \\
&\overset{(\text{CR})}{=} (\operatorname{vol}(B^n))^{-1} \int_{B^n} \nabla_y \Im(f(x+iy))\,dy \\
&\overset{(\text{Stokes})}{=} \operatorname{vol}(S^{n-1})/(\operatorname{vol}(B^n)) \int_{S^{n-1}} \Im(f(x+iy))\, y\,(dy) \\
&\overset{(\operatorname{vol}(S^{n-1})/(\operatorname{vol}(B^n)) = n)}{=} (n=)\ E_y[\Im(f(x+iy))\, y]\,.
\end{aligned}
$$

[23] We frequently appeal to Schmidt, Roux, and Bach 2011, Lem. 1.

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Example: worst function in the world

Consider the test function from Nesterov 2003, Ÿ 2.1.2

$$f_n(x) = L \left\{ \frac{1}{2} (x^{(1)})^2 + \sum_{i=1}^{n-1} (x^{(i+1)} - x^{(i)})^2 + (x^{(n)})^2 - x^{(1)} \right\} \qquad (9)$$

for $x_1 = 0$, $L = 10^{-8}$, $L_1(f) = 4L$ and $(x^?)^{(i)} = 1 \quad i=(n+1)$ with $x^{(i)}$.

(ia) Suboptimality gap $f(x_K) - f^?$ for the test function (9).

(ib) Suboptimality gap $f(x_K) - f^?$ for the test function (9).

Figure: The single-point Complex-smoothing (CS) compared to the multi-point Gaussian smoothing (GS) (fd) method from Nesterov and Spokoiny 2017.

Example: strong convexity $f(x) = \frac{1}{2}\|x\|_2^2$

(a) $f(x_K) - f^* , n = 10^0$. (b) $f(x_K) - f^* , n = 10^2$. (c) $f(x_K) - f^* , n = 10^4$.

Figure: The single-point Complex-smoothing (CS) compared to the multi-point Gaussian smoothing (GS) (fd and cd) method from Nesterov and Spokoiny 2017, Eq. (55). The rate is for (GS) (cd).

Example: non-convex optimization

Consider a Rosenbrock optimization problem

$$\underset{x \in \bar{2}B^2}{\text{minimize}} \quad (1 - x^{(1)})^2 + 100 \, (x^{(2)} - (x^{(1)})^2)^2 : \tag{10}$$

with $x^? = (1 ; 1)$.

(a) Suboptimality gap $f(x_K) - f^?$ for (10).

(b) Paths taken corresponding to Figure 4a.

Figure: The single-point Complex-smoothing (CS) method versus Gaussian-smoothing Nesterov and Spokoiny 2017.

What about noise?

We can handle[26] simulation noise, e.g., = (f (z)) + , z 2 2 C^n, (0; ^{2}).

Oracle (cs, noisy): g (x) = $\frac{n}{}$ = (f (x + i y)) y + $\frac{n}{}$ y; y $S^{n\ 1}$: (11)

Handling (n =) non-trivial. In general, we need $_k$ = O(1=k) and $_k$ = o($_k$).

[26] Jongeneel 2021.

[27] Building upon Hazan, Rakhlin, and Bartlett 2008; Akhavan, Pontil, and Tsybakov 2020.

[28] Shown by building upon Shamir 2013.

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Non-asymptotic results[27] for: constrained/unconstrained strongly convex functions and some non-convex functions (locally).

The algorithm is rate-optimal in the quadratic setting[28] (not surprising).

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What about noise?

We can handle[26] simulation noise, e.g., $= (f(z)) + \quad , z \; 2 \quad 2 \; C^n , \qquad (0; \; ^2)$.

$$\text{Oracle (cs, noisy):} \quad g(x) = \frac{n}{-} = (f(x + i y)) y + \frac{n}{-} y; \quad y \quad S^{n \; 1}: \qquad (11)$$

Handling ($n =$) non-trivial. In general, we need $_k = O(1=k)$ and $_k = o(\ _k)$.

Non-asymptotic results[27] for: constrained/unconstrained strongly convex functions and some non-convex functions (locally).

The algorithm is rate-optimal in the quadratic setting[28] (not surprising).

Why the ball B^n and not some other geometry
$M \; 2 \; M = f M \quad [\ 1; 1]^n : M$ di eomorphic to B^n g? Optimal in the sense that

$$\min_{M 2 M} \frac{\text{vol}(\ @M)}{\text{vol}(\ M)} = \frac{n}{-}; \quad B^n = \operatorname{argmin}_{M 2 M} \frac{\text{vol}(\ @M)}{\text{vol}(\ M)}; \qquad (12)$$

which follows from the isoperimetric inequality in R^n Osserman 1978.

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[26] Jongeneel 2021.

[27] Building upon Hazan, Rakhlin, and Bartlett 2008; Akhavan, Pontil, and Tsybakov 2020.

[28] Shown by building upon Shamir 2013.

Example: non-convex optimization (outlook)

Regularity of ODE/PDE constrained optimization problems can often be understood. We apply our zeroth-order algorithm to a ODE problem[29].

(a) Decay of $f(x_K)$ with the total number K of iterations for 10 independent simulation runs.

(b) Trajectories starting from $\grave{}(0)$ (unknown), x_0 (initial) and x_K for $K = 10^5$ (optimized).

Figure: Estimating the initial state $\grave{}(0)$ of a Lorenz system from a noisy measurement of the state $\grave{}(2) = '^2(\grave{}(0))$ (grey circle in 5b) at time 2. Even though the initial estimate x_0 is close to the optimized estimate x_K, $'^2(x_0)$ is far from $'^2(\grave{}(0))$.

[29] The complex-step derivative is implemented in an airfoil optimization package. Their underlying algorithm relies on sequential quadratic programming Nocedal and Wright 2006, Ch. 18, as such, the guarantees one can provide are different, see https://mdolab-cmplxfoil.readthedocs-hosted.com/en/latest/index.html , our work aims at providing rigorous guarantees with respect to the optimization algorithm itself.

The end

Main take away: single-point estimator where $_k = O(1=k)$ can be safely implemented.

Many open problems remain.

contact: wjongeneel.nl (slides will appear there).

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Appendix.

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